Shivaji University, Kolhapur
Question Bank For March 2022 (Summer) Examination

Sub Code:81662 **Subject Name: Metric Spaces**

Question Bank

Multiple choice questions

1. If ϱ_1 and ϱ_2 are metric on M, then which of the following is not a metric no M?

   a) $\varrho_1 + \varrho_2$ b) $\varrho_1 - \varrho_2$ c) $\frac{\varrho_1}{2}$ d) $\frac{\varrho_1 + \varrho_2}{2}$

2. Consider the following statements.

   I) Let $\langle M, \varrho \rangle$ be a metric space. For any $A \subset M$ and $\sigma = \varrho|_A$ i.e. σ is restriction of ϱ to A, then $\langle A, \sigma \rangle$ forms a metric space.

   II) If $\langle M, \varrho_1 \rangle$ and $\langle M, \varrho_2 \rangle$ are metric spaces, then $\langle M, \varrho_1 + \varrho_2 \rangle$ is also a metric space.

   Then...

   a) only I) is true. b) only II) is true.
   c) both I)and II) are true. d) both I)and II) are false.

3. The set of real numbers with absolute value metric is a metric space, which is usually denoted by ...

   a) R^1 b) R^2 c) R_d d) R^∞

4. Let $\langle M_1, \varrho_1 \rangle$ and $\langle M_2, \varrho_2 \rangle$ be metric spaces and $f : M_1 \to M_2$. We say that function $f(x) \to L \in M_2$ as $x \to a \in M_1$ from the right, if given $\epsilon > 0$, $\exists\, \delta > 0$ such that ...

   a) $\varrho_2\,(f(x), L) < \epsilon,\ (0 < \varrho_1\,(x, a) < \delta)$ b) $\varrho_1\,(f(x), L) < \epsilon,\ (0 < \varrho_2\,(x, a) < \delta)$
   c) $\varrho_2\,(f(x), L) < \epsilon,\ (0 < \varrho_2\,(x, a) < \delta)$ d) $\varrho_1\,(f(x), L) < \epsilon,\ (0 < \varrho_1\,(x, a) < \delta)$

5. Let $\langle M_1, \varrho_1 \rangle$ and $\langle M_2, \varrho_2 \rangle$ be metric spaces and $f : M_1 \to M_2$. We say that function $f(x)$ is continuous at $a \in M_1$ if ...

   a) $\lim_{x \to a} f(x) = a$ b) $\lim_{x \to a} f(x) \neq a$
   c) $\lim_{x \to a} f(x) \neq f(a)$ d) $\lim_{x \to a} f(x) = f(a)$

6. Let $\langle M, \varrho \rangle$ be a metric space and let $\{s_n\}$ be a sequence of points in M, we say that sequence $s_n \to L \in M$ as $n \to \infty$, if given $\epsilon > 0, \exists\, N \in I$ such that ...

   a) $\varrho(s_n, L) < \epsilon,\ \forall n \geq N.$
   b) $\varrho(s_n, L) < \epsilon,\ \forall n \in I.$
   c) $\varrho(s_n, L) > \epsilon,\ \forall n \geq N.$
   d) $\varrho(s_n, L) = \epsilon,\ \forall n \geq N.$

7. Consider the following statements.

   I) Every convergent sequence in any metric space is a Cauchy sequence.

   II) Every Cauchy sequence in any metric space is a convergent sequence.

   Then...

   a) only I) is true.
   b) only II) is true.
   c) both I)and II) are true.
   d) both I)and II) are false.

8. Which of the following is not a Cauchy sequence in a metric space R^1?

   a) $\{n\}$
   b) $\left\{\dfrac{n+4}{n}\right\}$
   c) $\left\{\left(\dfrac{1}{2}\right)^n\right\}$
   d) $\left\{\left(1+\dfrac{1}{n}\right)^n\right\}$

9. In a metric space $\langle M, \varrho \rangle$ with $M = [0, 1]$ and ϱ a usual metric defined by $\varrho(x, y) = |x - y|$, the open ball $B\left[\frac{1}{4}; \frac{1}{2}\right] = \ldots$

   a) $\left(-\frac{3}{4}, \frac{3}{4}\right)$
   b) $\left(0, \frac{3}{4}\right)$
   c) $\left[0, \frac{3}{4}\right)$
   d) $\left[-\frac{3}{4}, \frac{3}{4}\right]$

10. In a discrete metric space $M = R_d$, i.e the real line with discrete metric, $B[0; 1] = \ldots$

    a) $\{0\}$
    b) $\{1\}$
    c) R_d
    d) ϕ

11. In a discrete metric space $M = R_d$, i.e the real line with discrete metric, for any $a \in (0, 1)$, $B\left[a; \frac{1}{a}\right] = \ldots$

    a) $\{a\}$
    b) $\left\{\frac{1}{a}\right\}$
    c) R_d
    d) ϕ

12. For any $a, b \in R^1$ with $a < b$, which of the following is an open set in R^1?

    a) $[a, b)$
    b) $[a, b]$
    c) $\{a\}$
    d) (a, b)

13. In a metric space intersection of an infinite number of open sets is ....

    a) need not be an open set
    b) always an open set
    c) is closed set
    d) neither open nor closed set

14. Every subset of discrete metric space R_d is ....

a) both open and closed in R_d. b) open but not closed in R_d.
c) closed and not open in R_d. d) neither open nor closed in R_d.

15. Consider the following statements.

I) If E is any subset of metric space M then $E \subset \bar{E}$.

II)If E is any subset of metric space M then E is closed subset of M if $E = \bar{E}$.

Then...

a) only I) is true. b) only II) is true.
c) both I)and II) are true. d) both I)and II) are false.

16. In any metric space $\langle M, \varrho \rangle$, M and ϕ are...

a) open but not closed b) closed but not open
c) neither open nor closed d) both open and closed

17. In a metric space union of an infinite number of closed sets is ....

a) need not be a closed set b) always a closed set
c) is open set d) neither open nor closed set

18. If $f : R^1 \to R^1$ defined by $f(x) = x - 1$, then the inverse image of the open set $(0, 1)$ is ...

a) $(1, 2)$ b) $[1, 2)$ c) $(1, 2]$ d) $[1, 2]$

19. Consider the following statements.

I) There exists a suset A of a metric space R_d such that $\bar{A} = R_d$.

II)There exists a suset A of a metric space R^1 such that $\bar{A} = R^1$.

Then...

a) only I) is true. b) only II) is true.
c) both I)and II) are true. d) both I)and II) are false.

20. Which of the following is not a closed subset of R^1?

a) $\{a\}$ b) $(-\infty, a)$ c) $(-\infty, \infty)$ d) $[a, \infty)$

21. Let $\langle M, \varrho \rangle$ be any metric space and let A be any nonempty subset of M. If $a \in A$ and $B_A[a; r] = \{x \in A | \varrho(a, x) < r\}$, $B_M[a; r] = \{x \in M | \varrho(a, x) < r\}$ then ....

a) $B_M[a;r] = A \cap B_A[a;r]$
b) $B_A[a;r] = A \cup B_M[a;r]$
c) $B_M[a;r] = A \cup B_A[a;r]$
d) $B_A[a;r] = A \cap B_M[a;r]$

22. If M is connected metric space then. . . .

a) M has a proper subset which is both open and closed.
b) M has no proper subset which is both open and closed.
c) M is not open.
d) M is not closed.

23. In a usual metric space R^1, the set $A = (0, 1] \cup [1, 2]$ is . . .

a) an open set in R^1.
b) a closed set in R^1.
c) a connected set in R^1.
d) compact set in R^1.

24. If χ is a continuous characteristic function on a connected metric space M, then . . .

a) $\chi(x) = c, \forall x \in M$ where $c \in \{0, 1\}$.
b) $\chi(x) = 0, \forall x \in M$.
c) $\chi(x) = 1, \forall x \in M$.
d) $\chi(x) = c, \forall x \in M$ and $c \notin \{0, 1\}$.

25. If A is not a connected subset of R^1 then . . .

a) A may be a singleton set.
b) A may be an interval.
c) A may be union of intervals with nonempty intersection.
d) A may be union of intervals with empty intersection.

26. Consider the following statements.

I) If A is any connected subset of metric space M, then $\bar{A}$ is also connected.

II)If A, B are any connected subset of metric space M and $A \subset C \subset B$, then C is also connected.

Then. . .

a) only I) is true.
b) only II) is true.
c) both I)and II) are true.
d) both I)and II) are false.

27. If $A = (0, \infty) \subset R_d$, then $\text{diam}(A) = \dots$

a) 0
b) 1
c) ∞
d) c, where $c \in (1, \infty)$

28. For any $a, b, c \in R$, which of the following subset of metric space R^1 has a diameter different from $b - a$?

   a) $(a, b]$ b) (a, b) c) $[a + c, b + c]$ ✓d) $[ac, bc]$

29. Consider the following statements.

   I) Every totally bounded set is bounded.

   II)Every bounded set is totally bounded.

   Then...

   ✓a) only I) is true. b) only II) is true.

   c) both I)and II) are true. d) both I)and II) are false.

30. The statement that "If $\langle M, \varrho \rangle$ is a complete metric space and if T is a contraction on M, then there is one and only one point $x \in M$ such that $Tx = x$ " is called ....

   ✓a) Picard fixed point theorem b) Nested Interval theorem

   c) Picard contraction theorem d) Picard completeness theorem

31. Which of the following condition is satisfied by a contraction operator ϱ on a metric space $\langle M, \varrho \rangle$?

   a) $\varrho(Tx, Ty) \leq \frac{3}{2}\varrho(x, y)$, $\forall x, y \in M$ ✓b) $\varrho(Tx, Ty) \leq \frac{1}{2}\varrho(x, y)$, $\forall x, y \in M$

   c) $\varrho(Tx, Ty) \leq \varrho(x, y)$, $\forall x, y \in M$ d) $\varrho(Tx, Ty) = \varrho(x, y)$, $\forall x, y \in M$

32. If T is contraction mapping on metric space M then ....

   a) T is decreasing b) T is increasing

   ✓c) T is continuous d) T is constant

33. The statement that "If $\langle M, \varrho \rangle$ is any complete metric space and for each $n \in I$, F_n is a closed bounded subset of M such that

   (a) $F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \supset \cdots$,
   and

   (b) diam $F_n \to 0$ as $n \to \infty$,

   then $\bigcap_{n=1}^{\infty} F_n$ contains precisely one point." is called ....

   a) Picard fixed point theorem ✓b) Generalized Nested Interval theorem

   c) Picard contraction theorem d) Picard completeness theorem

34. The metric space $[a, b]$ with absolute value metric is ....

a) complete but not totally bounded
b) totally bounded but not complete
c) both complete and totally bounded
d) neither complete nor totally bounded

35. If a metric space $\langle M, \varrho \rangle$ has Heine - Borel property, then M is ...

a) both complete and totally bounded
b) neither complete nor totally bounded
c) complete but not totally bounded
d) totally bounded but not complete

36. Consider the following statements.

I) If a metric space $\langle M, \varrho \rangle$ is compact then M has the Heine-Borel property.

II) If a metric space $\langle M, \varrho \rangle$ has Heine - Borel property, then $\langle M, \varrho \rangle$ is compact.

Then...

a) only I) is true.
b) only II) is true.
c) both I)and II) are true.
d) both I)and II) are false.

37. If A is a closed subset of the compact metric space $\langle M, \varrho \rangle$, then the metric space $\langle A, \varrho \rangle$ is ...

a) both complete and totally bounded
b) neither complete nor totally bounded
c) complete but not totally bounded
d) totally bounded but not complete

38. The family of open intervals $\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ $n = 3, 4, 5, \ldots$ is an open covering of ...

a) the metric space $(0, 1)$ with absolute value metric
b) the metric space $[0, 1]$ with absolute value metric
c) the metric space $(0, 1]$ with absolute value metric
d) the metric space $[0, 1)$ with absolute value metric

39. If a real valued function f is continuous on the closed bounded interval $[a, b]$ then ...

a) there exists atleast one point $x \in M$ such that f attains its minimum value at x.
b) there exists only one point $x \in M$ such that f attains its minimum value at x.
c) there exists atmost one point $x \in M$ such that f attains its minimum value at x.
d) None of these

40. If a real valued function f is continuous on the compact metric space M then ...

a) there exists atleast one point $x \in M$ such that f attains its maximum value at x.

b) there exists only one point $x \in M$ such that f attains its maximum value at x.

c) there exists atmost one point $x \in M$ such that f attains its maximum value at x.

d) None of these

Questions for 8 Marks

1. Define Metric space. Show that the function $\varrho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varrho(x, y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{\frac{1}{2}}, \; x, y \in \mathbb{R}^n,$$

where $\mathbb{R}^n = \{x = \langle x_1, x_2, \cdots, x_n\rangle : x_k \in \mathbb{R}, \; k = 1, 2, \cdots, n\}$, forms a metric on $\mathbb{R}^n$.

2. Let $\langle M, \varrho\rangle$ be a Metric space and a be any point in M, also let f and g be any real valued functions whose domains are subsets of M. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = N$, then show that

(a) $\lim_{x\to a} [f(x) + g(x)] = L + N$

(b) $\lim_{x\to a} [f(x) \cdot g(x)] = L \cdot N$

3. Define open ball $B[x; r]$ in a metric space $\langle M, \varrho\rangle$. Show that any open ball in a metric space $\langle M, \varrho\rangle$ is an open subset of M.

4. Let $\langle M_1, \varrho_1\rangle$ and $\langle M_2, \varrho_2\rangle$ be metric spaces and let $f : M_1 \to M_2$. Show that f is continuous on M_1 if and only if $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

5. Define limit point. Show that for any subset E of metric space $\langle M, \varrho\rangle$, $x \in M$ is a limit point of E if and only if every open ball $B[x; r]$ about x contains at least one point of E.

6. Let $\langle M_1, \varrho_1\rangle$ and $\langle M_2, \varrho_2\rangle$ be metric spaces and let $f : M_1 \to M_2$. Show that f is continuous on M_1 if and only if $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.

7. Define open set and closed set. Show that in any metric space $\langle M, \varrho\rangle$ complement of open set is closed set and that of closed set is an open set.

8. Define closure of a set in a metric space. Show that in any metric space $\langle M, \varrho\rangle$, for any subset E of M, its closure $\bar{E}$ is closed set in M.

9. Show that every totally bounded subset of a metric space is bounded.

10. In a metric space $\langle M, \varrho \rangle$, for any proper subset A of M show that the subset G_A of A is an open set in $\langle A, \varrho \rangle$ if and only if there exists an open subset G_M of $\langle M, \varrho \rangle$ such that $G_A = A \cap G_M$.

11. Let $\langle M, \varrho \rangle$ be any metric space. Show that M is connected if and only if every continuous characteristic function on M is constant.

12. If $\langle M, \varrho \rangle$ is any complete metric space and T is a contraction on M, show that there is one and only one point x in M such that $Tx = x$.

13. If $\langle M, \varrho \rangle$ is any complete metric space and for each $n \in I$, F_n is a closed bounded subset of M such that

    (a) $F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \supset \cdots$,
    and

    (b) diam $F_n \to 0$ as $n \to \infty$,

    then show that $\bigcap_{n=1}^{\infty} F_n$ contains precisely one point.

14. Define comapact metric space. Show that the metric space $\langle M, \varrho \rangle$ is compact if and only if every sequence of points in M has a subsequence converging to a point in M.

15. Show that the metric space $\langle M, \varrho \rangle$ is compact if and only if, whenever $\mathfrak{F}$ is a family of closed subsets of M with the finite intersection property, then $\bigcap_{F \in \mathfrak{F}} F \neq \phi$.

Questions for 4 Marks

1. If ϱ and σ are metric on M then show that $\varrho + \sigma$ is also a metric on M.

2. If ϱ is a metric on M and $k > 0$ then show that $k\varrho$ is also a metric on M.

3. Show that any Cauchy sequence in a metric space R_d is convegent.

4. If $\{x_n\}$ is a convergent sequence in a metric space R_d then show that there exists $N \in I$ such that
$$x_N = x_{N+1} = x_{N+2} = \cdots .$$

5. Show that the function $\varrho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by
$$\varrho(x, y) = |x - y|, \; x, y \in \mathbb{R},$$
forms a metric on $\mathbb{R}$.

6. Show that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by
$$d(x, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y, \end{cases}$$
forms a metric on $\mathbb{R}$.

7. Show that every open interval in R^1 is an open set.
8. Show that arbitrary union of open subsets of a metric space M is also an open subset of M.
9. Show that finite intersection of open subsets of a metric space M is also an open subset of M.
10. Show that every subset of a metric space R_d is an open set.
11. Show that arbitrary intersection of closed subsets of a metric space M is a closed subset of M.
12. Show that finite union of closed subsets of a metric space M is a closed subset of M.
13. If G is an open set in a metric space M then show that G' is closed.
14. If F is a closed set in a metric space M then show that F' is open.
15. Show that any finite subset of a metric space is closed.
16. Show that the metric spaces R^1 and R_d are not homeomorphic.
17. If f is a continuous function from a connected metric space M_1 into a metric space M_2 then show that the range of f is also connected.
18. Giving an example prove that the union of two connected subsets of a metric space need not be connected.
19. Prove that if A is connected subset of a metric space M then $\bar{A}$ is also connected.
20. If A is totally bounded subset of a metric space R_d then show that A contains finite number of points.
21. Show that every finte subset of R_d is totally bounded.
22. Giving an example of an infinite subset of metric space l^2, prove that every bounded set need not be totally bounded.
23. If A is a closed subset of complete metric space $\langle M, \varrho \rangle$ then show that $\langle A, \varrho \rangle$ is complete.
24. Show that any contraction operator on a meric space M is continuous.
25. Show that the operator T on $[0, \frac{1}{3}]$ defined by

$$T(x) = x^2, \;\; 0 \leq x \leq \frac{1}{3},$$

    is contraction.
26. If A is a closed subset of compact metric space $\langle M, \varrho \rangle$ then show that $\langle A, \varrho \rangle$ is also compact.

27. If $\langle A, \varrho \rangle$ is a compact metric space, where A is subset of a metric space $\langle M, \varrho \rangle$ then show that A is closed subset of $\langle M, \varrho \rangle$.

28. If f is a continuous function from the compact metric space M_1 into a metric space M_2 then show that the range $f(M_1)$ of f is also compact.

29. If the real valued function f is continuous on the compact metric space M show that f attains its maximum value at some point of M.

30. Giving an example show that every connected subset of R^1 need not be compact.

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