FUNCTIONS OF COMPLEX VARIABLE

Functions of Complex Variable:

f(z) is a function of a complex variable z and is denoted by w.

i.e. w = f(z)

Where w = u+iv and u is a real part & v is a imaginary part.

Differentiability:

Let f(z) be a single valued function of the variable z, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

Provided that the limit exists and independent of the path along which $\delta z \to 0$.

Analytic Function:

A function f(z) is said to be analytic at a point z_0 if it is differentiable not only at z_0 but at every point of some neighborhood of z_0 .

Necessary condition for f(z) to be analytic:

The necessary condition for a function f(z) = u+iv to be analytic at all points in a region R are

i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

ii) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $provided$ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exists.

Note: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are known as **Cauchy Riemann equations**

SUFFICIENT CONDITION FOR F(Z) TO BE ANALYTIC:

Sufficient condition for f(z) to be analytic:

The sufficient condition for a function f(z) = u+iv to be analytic at all points in a region R are

i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

ii) $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ are continuous functions of x and y in region R.

Note: i) $\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h,0)-u(0,0)}{h}$

ii) $\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0,0+k)-u(0,0)}{k}$

iii) $\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h,0)-v(0,0)}{h}$

iv) $\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,0+k)-v(0,0)}{k}$

C-R EQUATIONS IN POLAR FORM:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Derivative of w in polar form:

$$\frac{dw}{dz} = (cos\theta - isin\theta)\frac{\partial w}{\partial r}$$

$$\frac{dw}{dz} = -\frac{i}{r}(cos\theta - isin\theta)\frac{\partial w}{\partial \theta}$$

Harmonic function:

Any function which satisfies Laplace's equation is known as harmonic function.

Laplace's equations:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

MILNE THOMSON METHOD (To construct an Analytic function)

Working rule:

Case 1.When u is given

Step1. Find $\frac{\partial u}{\partial x}$ and equate it to $\emptyset_1(x, y)$.

Step2. Find $\frac{\partial u}{\partial y}$ and equate it to $\emptyset_2(x, y)$.

Step3. Replace x by z and y by 0 in $\emptyset_1(x, y)$ to get$\emptyset_1(z, 0)$.

Step4. Replace x by z and y by 0 in $\emptyset_2(x, y)$ to get$\emptyset_2(z, 0)$.

Step5. Find f(z) by the formula $f(z) = \int\{\emptyset_1(z, 0) - i\emptyset_2(z, 0)\}dz + c$

Case II. When v is given

Step1. Find $\frac{\partial v}{\partial x}$ and equate it to $\varphi_2(x, y)$.

Step2. Find $\frac{\partial v}{\partial y}$ and equate it to $\varphi_1(x, y)$.

Step3. Replace x by z and y by 0 in $\varphi_1(x, y)$ to get $\varphi_1(z, 0)$.

Step4. Replace x by z and y by 0 in $\varphi_2(x, y)$ to get $\varphi_2(z, 0)$.

Step5. Find f(z) by the formula $f(z) = \int\{\varphi_1(z, 0) + i\varphi_2(z, 0)\}dz + c$

Case III. When u – v is given

We know that

$$f(z) = u + iv \dots\dots\dots\dots (1)$$

$$if(z) = iu - v \dots\dots\dots\dots (2)$$

Adding (1) & (2) we get

$$(1 + i)f(z) = (u - v) + i(u + v)$$

$$\Rightarrow F(z) = U + iV \qquad\qquad U = (u - v)$$

Where $F(z) = (1 + i)f(z) \dots\dots\dots\dots.. (3)$ $\qquad V = (u + v)$

Here $U = (u - v)$ is given. Find F(z) by using case I, then from (3) find

$$f(z) = \frac{F(z)}{1+i}$$

Case IV. When u + v is given

We know that

$$f(z) = u + iv \dots\dots\dots\dots (1)$$

$$if(z) = iu - v \dots\dots\dots\dots (2)$$

Adding (1) & (2) we get

$$(1+i)f(z) = (u-v) + i(u+v)$$

$$=> F(z) = U + iV \qquad\qquad U = (u-v)$$

Where $F(z) = (1+i)f(z) \dots\dots\dots\dots.. (3)$ $\qquad V = (u+v)$

Here $V = (u+v)$ is given. Find F(z) by using case II, then from (3) find

$$f(z) = \frac{F(z)}{1+i}$$

Complex Integration

Evaluation of line integral:

If $f(z) = w = u(x, y) + iv(x, y), then\ since\ dz = dx + idy$.

We have $\int_c f(z)dz = \int_c wdz$

$$= \int_c (u + iv)(dx + idy)$$

$$= \int_c (udx - vdy) + i \int_c (vdx + udy)$$

This shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Cauchy's Integral Theorem:

If a function f(z) is analytic and its derivative f '(z) continuous at all points inside and on a simple closed curve c, then $\int_c f(z)dz = 0$.

Cauchy's Integral Formula:

If f(z) is analytic within and on a closed curve c, and if a is any point within c, then

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

In general,

$$\int_c \frac{f(z)dz}{(z-a)^{n+1}} = \frac{1}{n!}\,2\pi i f^{n}(a)$$

The Calculus of Residues:

Zero of Analytic Function:

A zero of analytic function $f(z)$ is the value of z for which $f(z) = 0$.

Singular Point: A point at which a function $f(z)$ is not analytic is known as a singular point or singularity of the function.

Isolated singular Point:

If $z = a$ is a singularity of $f(z)$ and there is no other singularity within a small circle surrounding the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$; otherwise it is called non-isolated.

For example, the function $\frac{1}{(z-1)(z-3)}$ has two isolated singular points, namely $z = 1$ and $z = 3$.

Example of non-isolated singularity:

The function $\frac{1}{sin\frac{\pi}{z}}$ is not analytic at the points where $sin\frac{\pi}{z} = 0$ i. e. at the points $\frac{\pi}{z} = n\pi$ $i.e.$ at the points $z = \frac{1}{n}(n = 1,2,3, \ldots..)$. Thus $z = 1, \frac{1}{2}, \frac{1}{3}, \ldots. = 0$

Are the points of singularity. $z = 0$ is the non isolated singularity of the function $\frac{1}{sin\frac{\pi}{z}}$ because in the neighborhood of $z = 0$, there are infinite many other singularities.

Pole of order m: Let a function $f(z)$ have an isolated singular point $z = a$, $f(z)$

Can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{1}{(z-a)^m}\{b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + \cdots + b_m\}$$

Then $z = a$ is said to be a pole of order m of the function $f(z)$, when $m = 1$ the pole is said to be simple pole.

Method of finding Residues

(a) Residue at simple pole :

If $f(z)$ has a simple pole at $z = a$, then

$$Res(at\ z = a) = \lim_{z \to a} (z - a) f(z)$$

(b) Residue at a pole of order n:

If $f(z)$ has a pole of order n at $z = a$, then

$$Res(at\ z = a) = \lim_{z \to a} \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)] \right\}$$

(c) Residue Theorem:

If $f(z)$ is analytic in a closed curve C, except at a finite number of poles within C, then

$$\int_c f(z)\, dz = 2\pi i (sum\ of\ residues\ at\ the\ poles\ within\ C)$$

EVALUTION OF REAL DEFINITE INTEGRALS BY CONTOUR INTEGRATION

(a) INTEGRATION ROUND UNIT CIRCLE OF THE TYPE

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

Where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$

Method:-

Convert $\sin\theta$, $\cos\theta$ into z

Consider a circle of unit radius with centre at origin, as contour.

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left[z - \frac{1}{z}\right], \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left[z + \frac{1}{z}\right]$$

Where $z = re^{i\theta} = 1.e^{i\theta} = e^{i\theta}$

As we know

$$z = e^{i\theta}, dz = e^{i\theta} i d\theta = zi d\theta \text{ or } d\theta = \frac{dz}{iz}$$

The integrand is converted into a function of z. Then apply Cauchy's residue theorem to evaluate the integral.

(b) Evaluation of $\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} dx$ where $f_1(x)$ and $f_2(x)$ are polynomials in x.

Such integrals can be reduced to contour integrals, if

(i) $f_2(x)$ has no real roots.

(ii) The degree of $f_2(x)$ is greater than that of $f_1(x)$ by at least two.

Procedure: Let $f(x) = \frac{f_1(x)}{f_2(x)}$

Consider $\int_c f(z)dz$ where C is a curve, consisting of upper half C_R of the circle $|z| = R$ and part of real axis from $-R \; to \; R$.

If there are no poles of $f(z)$ on the real axis, the circle $|z| = R$ which is arbitrary can be taken such that there is no singularity on its circumference C_R in the upper half of the plane, but possibly some poles inside the contour C specified above.

Using Cauchy's theorem of residues, we have

$$\int_c f(z)dz = 2\pi i(sum \; of \; the \; residues \; of \; f(z) at \; the \; poles \; within \; C)$$