DR. J. J. MAGDUM COLLEGE OF ENGINEERING, JAYSINGPUR

" FOURIER SERIES"

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By

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** FOURIER SERIES **

Introduction: In many physical and engineering problems, particularly those connected with vibration and conduction of heat, it is more useful to be able to express a real valued function in series of sines and cosines. Most of the single valued periodic functions which occur in applied mathematics can be expressed in the form

$$\begin{aligned}\frac{a_0}{2} + a_1 cosx + a_2 cos2x + ... + a_n cosnx + ... + \\ + b_1 sinx + b_2 sin2x + ... + b_n sinnx + ... \\ = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx\end{aligned}$$

is called the Fourier series, where $a_0, a_1, a_2, ..., a_n, ..., b_1, b_2, ..., b_n, ...$ are constants.

** USEFUL INTEGRALS **

The following integrals are useful in Fourier Series.

$(i)\int_0^{2\pi} sinnxdx = 0,$ $(ii)\int_0^{2\pi} cosnxdx = 0,$

$(iii)\int_0^{2\pi} sin^2 nxdx = \pi,$ $(iv)\int_0^{2\pi} cos^2 nxdx = \pi,$

$(v)\int_0^{2\pi} sinnx \cdot sinmxdx = 0,$ $(vi)\int_0^{2\pi} cosnx \cdot cosmxdx = 0,$

$(vii)\int_0^{2\pi} sinnx \cdot cosmxdx = 0,$ $(viii)\int_0^{2\pi} sinnx \cdot cosnxdx = 0,$

$(ix) sinn\pi = 0, \quad cosn\pi = (-1)^n$ where $n \in I$

$(x)\int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$

$(xi)\int e^{ax} sinbxdx = \frac{e^{ax}}{a^2+b^2}(asinbx - bcosbx)$

$(xii)\int e^{ax} cosbxdx = \frac{e^{ax}}{a^2+b^2}(acosbx + bsinbx)$

Determination of Fourier Coefficients: (Euler's Formulae)

Let

$$f(x) = \frac{a_0}{2} + a_1 cosx + a_2 cos2x + ... + a_n cosnx + ... + \\ + b_1 sinx + b_2 sin2x + ... + b_n sinnx + ... \qquad (1)$$

(i) **To find a_0 :** Integrate both sides of (1) from $x = \alpha$ to $x = \alpha + 2\pi$

$$\int_{\alpha}^{\alpha+2\pi} f(x)dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} dx + a_1\int_{\alpha}^{\alpha+2\pi} cosxdx + \\ + a_2\int_{\alpha}^{\alpha+2\pi} cos2xdx + ... + a_n\int_{\alpha}^{\alpha+2\pi} cosnxdx + ... + \\ + b_1\int_{\alpha}^{\alpha+2\pi} sinxdx + b_2\int_{\alpha}^{\alpha+2\pi} sin2xdx + ... + \\ + b_n\int_{\alpha}^{\alpha+2\pi} sinnxdx + ...$$

$$\int_{\alpha}^{\alpha+2\pi} f(x)dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} dx \quad (\text{other int.} = 0 \text{ by } (i) \text{ and } (ii))$$

$$= \frac{a_0}{2} 2\pi \qquad \Rightarrow a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx$$

To find a_n: multiply each side of (1) by $cosnx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$

$$\int_{\alpha}^{\alpha+2\pi} f(x)cosnxdx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} cosnxdx + a_1\int_{\alpha}^{\alpha+2\pi} cosnxcosxdx +$$

$$+ a_2\int_{\alpha}^{\alpha+2\pi} cosnxcos2xdx + ... + a_n\int_{\alpha}^{\alpha+2\pi} cos^2 nxdx + ... +$$

$$+ b_1\int_{\alpha}^{\alpha+2\pi} cosnxsinxdx + b_2\int_{\alpha}^{\alpha+2\pi} cosnxsin2xdx + ... +$$

$$+ b_n\int_{\alpha}^{\alpha+2\pi} cosnxsinnxdx + ...$$

$$\int_{\alpha}^{\alpha+2\pi} f(x)cosnxdx = a_n \int_{\alpha}^{\alpha+2\pi} cos^2 nxdx$$

$$= a_n\pi \quad \text{(other int.=0 by above formulae)}$$

$$\therefore \quad a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)cosnxdx$$

By taking $n = 1, 2, 3, ...$ we can find the values of $a_1, a_2, ...$

To find b_n: multiply each side of (1) by *sinnx* and integrate from $x = \alpha$ to $x = \alpha + 2\pi$

$$\int_{\alpha}^{\alpha+2\pi} f(x)sinnxdx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} sinnxdx + a_1\int_{\alpha}^{\alpha+2\pi} sinnxcosxdx +$$

$$+ a_2\int_{\alpha}^{\alpha+2\pi} sinnxcos2xdx + ... + a_n\int_{\alpha}^{\alpha+2\pi} sinnxcosnxdx + ...+$$

$$+ b_1\int_{\alpha}^{\alpha+2\pi} sinnxsinxdx + b_2\int_{\alpha}^{\alpha+2\pi} sinnxsin2xdx + ...+$$

$$+ b_n\int_{\alpha}^{\alpha+2\pi} sin^2 nxdx + ...$$

$$\int_{\alpha}^{\alpha+2\pi} f(x) sinnxdx = b_n \int_{\alpha}^{\alpha+2\pi} sin^2 nxdx$$

$$= b_n \pi \quad \text{(other int.=0 by above formulae)}$$

$$\therefore \quad b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) sinnxdx$$

By taking $n = 1, 2, 3, ...$ we can find the values of $b_1, b_2, ...$

Corol 1. When $\alpha = 0$, the interval becomes $0 \leq x \leq 2\pi$ and above formulae becomes

$$a_0 = \frac{1}{\pi} \int_{0}^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) cosnxdx$$

$$b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) sinnxdx$$

Corol 2.:

When $\alpha = -\pi$, the interval becomes $-\pi \leq x \leq \pi$ and above formulae becomes

$$\begin{aligned} a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx \\ b_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx \end{aligned} \tag{A}$$

Fourier Series in Change of Interval:

$$\begin{aligned} f(x) = \frac{a_0}{2} + a_1 cos(\frac{\pi x}{c}) + a_2 cos(\frac{2\pi x}{c}) + ... + a_n cos(\frac{n\pi x}{c}) + ... + \\ + b_1 sin(\frac{\pi x}{c}) + b_2 sin(\frac{2\pi x}{c}) + ... + b_n sin(\frac{n\pi x}{c}) + ... \end{aligned} \tag{1}$$

(i) **To find a_0 :** Integrate both sides of (1) from $x = \alpha$ to $x = \alpha + 2c$

$$\int_{\alpha}^{\alpha+2c} f(x)dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2c} dx + a_1\int_{\alpha}^{\alpha+2c} cos(\frac{\pi x}{c})dx +$$
$$+ a_2\int_{\alpha}^{\alpha+2c} cos(\frac{2\pi x}{c})dx + ... + a_n\int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})dx +$$
$$+ ... + b_1\int_{\alpha}^{\alpha+2c} sin(\frac{\pi x}{c})dx + b_2\int_{\alpha}^{\alpha+2c} sin(\frac{2\pi x}{c})dx +$$
$$+ ... + b_n\int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})dx + ...$$

$$\int_{\alpha}^{\alpha+2c} f(x)dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2c} dx \quad \text{(other int.} = 0 \text{ by } (i) \text{ and } (ii))$$
$$= \frac{a_0}{2}2c \qquad \Rightarrow a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)dx$$

To find a_n:

multiply each side of (1) by $cos(\frac{n\pi x}{c})$ and integrate from $x = \alpha$ to $x = \alpha + 2c$

$$\int_{\alpha}^{\alpha+2c} f(x)cos(\frac{n\pi x}{c})dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2c} cos(\frac{\pi x}{c})dx+$$

$$+ a_1\int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})cos(\frac{\pi x}{c})dx+$$

$$+ a_2\int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})cos(\frac{2\pi x}{c})dx + ...+$$

$$+ a_n\int_{\alpha}^{\alpha+2c} cos^2(\frac{n\pi x}{c})dx + ...+$$

$$+ b_1\int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})sin(\frac{\pi x}{c})dx+$$

$$+ b_2\int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})sin(\frac{2\pi x}{c})dx + ...+$$

$$+ b_n\int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})sin(\frac{n\pi x}{c})dx + ...$$

$$\int_{\alpha}^{\alpha+2c} f(x) cos(\frac{n\pi x}{c})dx = a_n \int_{\alpha}^{\alpha+2c} cos^2(\frac{n\pi x}{c})dx$$
$$= a_n c \quad \text{(other int.=0 by above formulae)}$$
$$\therefore \quad a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x) cos(\frac{n\pi x}{c})dx$$

To find b_n: multiply each side of (1) by $sin(\frac{n\pi x}{c})$ and integrate from $x = \alpha$ to $x = \alpha + 2c$

$$\int_{\alpha}^{\alpha+2c} f(x) sin(\frac{n\pi x}{c})dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})dx +$$
$$+ a_1 \int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c}) cos(\frac{\pi x}{c})dx +$$

$$+ a_2 \int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})cos(\frac{2\pi x}{c})dx + ...+$$

$$+ a_n \int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})cos(\frac{n\pi x}{c})dx + ...+$$

$$+ b_1 \int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})sin(\frac{\pi x}{c})dx+$$

$$+ b_2 \int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})sin(\frac{2\pi x}{c})dx + ...+$$

$$+ b_n \int_{\alpha}^{\alpha+2c} sin^2(\frac{n\pi x}{c})dx + ...$$

$$\int_{\alpha}^{\alpha+2c} f(x)sin(\frac{n\pi x}{c})dx = b_n \int_{\alpha}^{\alpha+2c} sin^2(\frac{n\pi x}{c})dx$$

$$= b_n c \quad \text{(other int.=0 by above formulae)}$$

$$\therefore \quad b_n = \frac{1}{c} \int_{\alpha}^{\alpha+2c} f(x)sin(\frac{n\pi x}{c})dx$$

Corol. 1:

When $\alpha = 0$, the interval becomes $0 \leq x \leq 2c$ and above formulae becomes

$$a_0 = \frac{1}{c}\int_0^{2c} f(x)dx$$

$$a_n = \frac{1}{c}\int_0^{2c} f(x)cos(\frac{n\pi x}{c})dx$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)sin(\frac{n\pi x}{c})dx$$

Corol. 2: When $\alpha = -c$, the interval becomes $-c \leq x \leq c$ and above formulae becomes

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)cos(\frac{n\pi x}{c})dx$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)sin(\frac{n\pi x}{c})dx$$

Even and Odd Functions:

When $f(x)$ posses certain symmetrical properties about origin, the determination of coefficients in its Fourier expansion becomes quite simple.

For the function $f(x)$ to be odd or even, it must be defined in an interval, where origin is the mid-point of the interval i. e. $f(x)$ must have interval $-\pi < x < \pi$.

Even Function: $f(x)$ is said to be an even function in $-\pi < x < \pi$, if $f(-x) = f(x)$.

Thus functions $x^2, cosx, xsinx$ are even functions in $-\pi < x < \pi$.

Graphically an even function is symmetric about y-axis. Thus for even function $f(x)$ defined in the interval $-\pi \leq x \leq \pi$, the Fourier coefficients a_0, a_n, b_n given in (A) are reduced as follows.

$$
\begin{aligned}
a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x)dx + \int_0^{\pi} f(-x)dx\right] \quad \text{(Since by integral theorem)}
\end{aligned}
$$

$$
\begin{aligned}
a_0 =& \frac{1}{\pi}\left[\int_0^{\pi} f(x)dx + \int_0^{\pi} f(x)dx\right] \quad (\text{Since} \quad f(-x) = f(x) \\
=& \frac{2}{\pi}\int_0^{\pi} f(x)dx
\end{aligned}
$$

and

$$
\begin{aligned}
a_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x)cosnxdx + \int_0^{\pi} f(-x)cosn(-x)dx\right] \ (\text{Since by int. theore} \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x)cosnxdx + \int_0^{\pi} f(x)cosnxdx\right] \quad (\text{Since} \quad f(-x) = f(x) \\
=& \frac{2}{\pi}\int_0^{\pi} f(x)cosnxdx
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
b_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sinnxdx \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x) sinnxdx + \int_0^{\pi} f(-x) sinn(-x)dx\right] \text{ (Since by int. theorer} \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x) sinnxdx - \int_0^{\pi} f(x) sinnxdx\right] \quad \text{(Since } f(-x) = f(x) \\
=& 0
\end{aligned}
$$

Odd Function: $f(x)$ is said to be an odd function in $-\pi < x < \pi$, if $f(-x) = -f(x)$. Thus functions $x^3, sinx, x^2 sinx$ etc are odd functions in $-\pi < x < \pi$. Graphically an odd function is symmetric about origin. Thus for odd function $f(x)$ defined in the interval $-\pi \leq x \leq \pi$, the Fourier coefficients a_0, a_n, b_n given in (A) are reduced as follows.

$$
\begin{aligned}
a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x)dx + \int_0^{\pi} f(-x)dx\right] \quad \text{(Since by integral theorem)}
\end{aligned}
$$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} f(x)dx - \int_0^{\pi} f(x)dx\right] \quad (\text{Since} \quad f(-x) = -f(x)$$
$$= 0$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$
$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)cosnxdx + \int_0^{\pi} f(-x)cosn(-x)dx\right] \; (\text{Since by int. theore}$$
$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)cosnxdx - \int_0^{\pi} f(x)cosnxdx\right] \quad (\text{Since} \quad f(-x) = -f(x)$$
$$= 0$$

Similarly,

$$
\begin{aligned}
b_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sinnxdx \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x) sinnxdx + \int_0^{\pi} f(-x) sinn(-x) dx\right] \text{ (Since by int. theorem} \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x) sinnxdx + \int_0^{\pi} f(x) sinnxdx\right] \quad \text{(Since } f(-x) = -f(x) \text{)} \\
=& \frac{2}{\pi}\int_0^{\pi} f(x) sinxdx
\end{aligned}
$$

Fourier series for Discontinuous functions: Let the function $f(x)$ be defined by

$$
\begin{aligned}
f(x) =& f_1(x), \qquad \alpha < x < x_0 \\
=& f_2(x), \qquad x_0 < x < \alpha + 2\pi
\end{aligned}
$$

where x_0 is the point of discontinuity in the interval $(\alpha,\ \alpha + 2\pi)$

In such cases also, we obtain the Fourier series for $f(x)$ in the usual way. The values of a_0, a_n, b_n are evaluated by

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)dx + \int_{x_0}^{\alpha+2\pi} f_2(x)dx\right]$$
$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)cosnxdx + \int_{x_0}^{\alpha+2\pi} f_2(x)cosnxdx\right]$$
$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)sinnxdx + \int_{x_0}^{\alpha+2\pi} f_2(x)sinnxdx\right]$$

If $x = x_0$ is the point of finite discontinuity, then the sum of the Fourier series

$$= \frac{1}{2}\left[\lim_{h\to 0} f(x_0 - h) + \lim_{h\to 0} f(x_0 + h)\right]$$
$$= \frac{1}{2}\left[f(x_0 - 0) + f(x_0 + 0)\right]$$

Sine Series:

If it be required to expand $f(x)$ as a sine series in $0 < x < c$; then we extend the function reflecting it in the origin, so that $f(-x) = -f(x)$.

Then the extended function is odd in $(-c, c)$ and the expansion will give the desired Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n sin \frac{n\pi x}{c}$$

where

$$b_n = \frac{2}{c} \int_0^c f(x) sin \frac{n\pi x}{c} dx$$

Cosine Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n cos \frac{n\pi x}{c}$$

where

$$a_0 = \frac{2}{c} \int_0^c f(x) dx \ , a_n = \frac{2}{c} \int_0^c f(x) cos \frac{n\pi x}{c} dx.$$

Ex. 1.

Find Fourier series for the function $f(x) = \frac{(\pi-x)^2}{4}$, $0 < x < 2\pi$.

Solution: Let

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx \ , \qquad (1)$$

where

$$\begin{aligned} a_0 =& \frac{1}{\pi}\int_0^{2\pi} f(x)dx \ , \\ =& \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}dx \ , \\ =& \frac{\pi^2}{6} \ . \end{aligned}$$

$$\begin{aligned} a_n =& \frac{1}{\pi}\int_0^{2\pi} f(x)cosnxdx \ , \\ =& \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}cosnxdx \ , \end{aligned}$$

$$=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2 cosnxdx \ ,$$

$$=\frac{1}{4\pi}\left[(\pi-x)^2\cdot(\frac{sinnx}{n})-2(\pi-x)(-1)(\frac{-cosnx}{n^2})+(-2)(\frac{-sinnx}{n^3})\right]_0^{2\pi} \ ,$$

$$=\frac{1}{4\pi}\left[\frac{2\pi}{n^2}+\frac{2\pi}{n^2}\right] \ ,$$

$$=\frac{1}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)sinnxdx \ ,$$

$$=\frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}sinnxdx \ ,$$

$$=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2 sinnxdx \ ,$$

$$=\frac{1}{4\pi}\left[(\pi-x)^2\cdot(\frac{-cosnx}{n})-2(\pi-x)(-1)(\frac{-sinnx}{n^2})+(-2)(\frac{cosnx}{n^3})\right]$$

$$=0$$

Putting the values of a_0, a_n and b_n in series (1), we obtain required Fourier series as

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{cosnx}{n^2}$$

Ex. 2 Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

Solution: Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx \ , \qquad (1)$$

where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} dx$$

$$
\begin{aligned}
\therefore a_0 =& \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi} \ , \\
a_n =& \frac{1}{\pi}\int_0^{2\pi} f(x)cosnxdx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}cosnxdx \ , \\
=& \frac{1}{\pi(n^2+1)}\left[e^{-x}(-cosnx + nsinnx)\right]_0^{2\pi} \ , \\
=& \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{n^2+1} \ . \\
b_n =& \frac{1}{\pi}\int_0^{2\pi} f(x)sinnxdx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}sinnxdx \ , \\
=& \frac{1}{\pi(n^2+1)}\left[e^{-x}(-sinnx - ncosnx)\right]_0^{2\pi} \ , \\
=& \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{n}{n^2+1} \ .
\end{aligned}
$$

Putting values of a_0, a_n and b_n in (1), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \left(\frac{1}{2}cosx + \frac{1}{5}cos2x + \frac{1}{10}cos3x + ...\right) + \right.$$
$$\left. + \left(\frac{1}{2}sinx + \frac{2}{5}sin2x + \frac{3}{10}sin3x + ...\right)\right\}.$$

Ex. 3 Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$.

Solution: Let

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx \ , \qquad (1)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = -\frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2) cosnx dx \ ,$$
$$= \frac{1}{\pi}\left[(x - x^2)\frac{sinnx}{n} - (1 - 2x)\cdot(-\frac{cosnx}{n^2}) + (-2)\cdot(-\frac{sinnx}{n^3})\right]_{-\pi}^{\pi}$$
$$= -\frac{4(-1)^n}{n^2} \ ,$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2) sinnx dx \ ,$$
$$= \frac{1}{\pi}\left[(x - x^2)\frac{(-cosnx)}{n} - (1 - 2x)\cdot(-\frac{sinnx}{n^2}) + (-2)\cdot(-\frac{cosnx}{n^3})\right]_{-\pi}^{\pi}$$
$$= -\frac{2(-1)^n}{n} \ ,$$

Putting values of a_0, a_n and b_n in (1), we get

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{cosx}{1^2} - \frac{cos2x}{2^2} + \frac{cos3x}{3^2} - \frac{cos4x}{4^2} + ...\right] +$$

$$+2\left[\frac{sinx}{1}-\frac{sin2x}{2}+\frac{sin3x}{3}-\frac{sin4x}{4}+...\right].$$

Ex. 4 Expand $f(x) = xsinx$ as a Fourier series in $[0, 2\pi]$.

Solution: Let

$$f(x)=\frac{a_o}{2}+\sum_{n=1}^{\infty}a_n cosnx+\sum_{n=1}^{\infty}b_n sinnx \ , \qquad (1)$$

where

$$a_0 =\frac{1}{\pi}\int_0^{2\pi} xsinxdx = \frac{1}{\pi}\Big[x(-cosx)-1\cdot(-sinx)\Big]_0^{2\pi} = -2,$$

$$a_n =\frac{1}{\pi}\int_0^{2\pi} xsinxcosnxdx = \frac{1}{2\pi}\int_0^{2\pi} x(2cosnxsinx)dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x\Big[sin(n+1)x - sin(n-1)x\Big]dx$$

$$=\frac{1}{2\pi}\left[x\left\{\frac{-cos(n+1)x}{n+1}+\frac{cos(n-1)x}{n-1}\right\}-\right]$$

$$+\left[-1\left\{\frac{-sin(n+1)x}{(n+1)^2}+\frac{sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi},$$

$$=\frac{1}{2\pi}\left[2\pi\left\{-\frac{cos2(n+1)\pi}{n+1}+\frac{cos2(n-1)\pi}{n-1}\right\}\right]=\frac{2}{n^2-1}\quad (n\neq 1).$$

when $n=1$,

$$\begin{aligned}a_1 =&\frac{1}{\pi}\int_0^{2\pi} xsinxcosxdx=\frac{1}{2\pi}\int_0^{2\pi} xsin2xdx\\ =&\left[x\left(\frac{-cos2x}{2}\right)-1\cdot\left(\frac{-sin2x}{4}\right)\right]_0^{2\pi}=-\frac{1}{2}\ .\\ b_n =&\frac{1}{\pi}\int_0^{2\pi} xsinxsinnxdx=\frac{1}{2\pi}\int_0^{2\pi} x(2sinxsinnx)dx\\ =&\frac{1}{2\pi}\int_0^{2\pi} x\Big[cos(n-1)x-cos(n+1)x\Big]dx\end{aligned}$$

$$
\begin{aligned}
b_n =& \frac{1}{2\pi}\left[x\left\{\frac{sin(n-1)x}{n-1}+\frac{sin(n+1)x}{n+1}\right\}-\right.\\
&\left.-1\left\{\frac{-cos(n-1)x}{(n-1)^2}+\frac{cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi},\\
=& \frac{1}{2\pi}\left[\left\{\frac{cos2(n-1)\pi}{(n-1)^2}-\frac{cos2(n+1)\pi}{(n+1)^2}-\frac{1}{(n-1)^2}+\frac{1}{(n+1)^2}\right\}\right]\\
=& 0 \quad (n \neq 1).
\end{aligned}
$$

when $n = 1$,

$$
\begin{aligned}
b_1 =& \frac{1}{\pi}\int_0^{2\pi} xsinxsinxdx = \frac{1}{2\pi}\int_0^{2\pi} x(1-cos2x)dx\\
=& \left[x\left(x-\frac{sin2x}{2}\right)-1\cdot\left(\frac{x^2}{2}+\frac{cos2x}{4}\right)\right]_0^{2\pi} = \pi
\end{aligned}
$$

Substituting the values of $a's$ and $b's$ in (1), we get

$$xsinx = -1 + \pi sinx - \frac{1}{2}cosx + \frac{2}{2^2-1}cos2x + \frac{2}{3^2-1}cos3x + ...$$

Ex. 5 Find the Fourier series expansion of $f(x) = 2x - x^2$ in $(0, 3)$ and hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... - \infty = \frac{\pi^2}{12}$$

Solution: Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{c} \text{ , where } 2c = 3. \quad (1)$$

where

$$a_0 = \frac{1}{c}\int_0^{2c} f(x)dx = \frac{2}{3}\int_0^3 (x - x^2)dx = \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = 0$$

$$
\begin{aligned}
a_n =& \frac{1}{c}\int_0^{2c} f(x) cos\frac{n\pi x}{c}dx = \frac{2}{3}\int_0^3 (2x - x^2) cos\frac{2n\pi x}{3}dx \\
=& \frac{2}{3}\Bigg[(2x - x^2)\frac{sin2n\pi x/3}{2n\pi/3} - (2 - 2x)\bigg(-\frac{cos2n\pi x/3}{(2n\pi/3)^2}\bigg) + \\
&+ (-2)\bigg(-\frac{sin2n\pi x/3}{(2n\pi/3)^3}\bigg)\Bigg]_0^3 = -\frac{9}{n^2\pi^2} \ , \\
b_n =& \frac{1}{c}\int_0^{2c} f(x) sin\frac{n\pi x}{c}dx = \frac{2}{3}\int_0^3 (2x - x^2) sin\frac{2n\pi x}{3}dx \\
=& \frac{2}{3}\Bigg[(2x - x^2)\frac{-cos2n\pi x/3}{2n\pi/3} - (2 - 2x)\bigg(-\frac{sin2n\pi x/3}{(2n\pi/3)^2}\bigg) + \\
&+ (-2)\bigg(\frac{cos2n\pi x/3}{(2n\pi/3)^3}\bigg)\Bigg]_0^3 = \frac{3}{n\pi} \ ,
\end{aligned}
$$

Substituting the values of a_0, a_n, b_n in (1), we get

$$2x - x^2 = -\sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} \frac{3}{n\pi} sin\frac{2n\pi x}{3}$$

Putting $x = 3/2$, we get

$$3 - \frac{9}{4} = -\sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} cos n\pi \quad or -\sum_{n=1}^{\infty} \frac{cos n\pi}{n^2} = \frac{\pi^2}{9} \cdot \frac{3}{4}$$

or

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... = \frac{\pi^2}{12}.$$

Ex. 6 Find the Fourier series for

$$\begin{aligned} f(t) =& 0 \quad , -2 < t < -1 \\ =& 1 + t \quad , -1 < t < 0 \\ =& 1 - t \quad , 0 < t < 1 \\ =& 0 \quad , 1 < t < 2 \end{aligned}$$

Solution:

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi t}{2} + \sum_{n=1}^{\infty} a_n sin\frac{n\pi t}{2} \tag{1}$$

where

$$\begin{aligned}
a_0 =& \frac{1}{2}\int_{-2}^{2} f(t)dt \\
=& \frac{1}{2}\left[\int_{-2}^{-1} 0dt + \int_{-1}^{0}(1+t)dt + \int_{0}^{1}(1-t)dt + \int_{1}^{2} 0dt\right] \\
=& \frac{1}{2}\left[\left(t+\frac{t^2}{2}\right)_{-1}^{0} + \left(t-\frac{t^2}{2}\right)_{0}^{1}\right] = \frac{1}{2} \\
a_n =& \frac{1}{2}\int_{-2}^{2} f(t)cos\frac{n\pi t}{2}dt \\
=& \frac{1}{2}\left[0 + \int_{-1}^{0}(1+t)cos\frac{n\pi t}{2}dt + \int_{0}^{1}(1-t)cos\frac{n\pi t}{2}dt + 0\right]
\end{aligned}$$

$$
\begin{aligned}
a_n =& \frac{1}{2}\left\{\left[(1+t)(sin\frac{n\pi t}{2})\cdot\frac{2}{n\pi} - (1)(-cos\frac{n\pi t}{2})\cdot(\frac{2}{n\pi})^2\right]_{-1}^{0} + \right.\\
& \left. + \left[(1-t)(sin\frac{n\pi t}{2})\cdot\frac{2}{n\pi} - (-1)(-cos\frac{n\pi t}{2})\cdot(\frac{2}{n\pi})^2\right]_{0}^{1}\right\} ,\\
=& \frac{1}{2}\left[\frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\right] ,\\
=& \frac{4}{n^2\pi^2}\left(1 - cos\frac{n\pi}{2}\right) ,\\
b_n =& \frac{1}{2}\int_{-2}^{2} f(t)sin\frac{n\pi t}{2}dt\\
=& \frac{1}{2}\left[0 + \int_{-1}^{0}(1+t)sin\frac{n\pi t}{2}dt + \int_{0}^{1}(1-t)sin\frac{n\pi t}{2}dt + 0\right]
\end{aligned}
$$

$$
\begin{aligned}
b_n =& \frac{1}{2}\left\{ \left[(1+t)(-cos\frac{n\pi t}{2}) \cdot \frac{2}{n\pi} - (1)(-sin\frac{n\pi t}{2}) \cdot (\frac{2}{n\pi})^2 \right]_{-1}^{0} + \right. \\
& \left. + \left[(1-t)(-cos\frac{n\pi t}{2}) \cdot \frac{2}{n\pi} - (-1)(-sin\frac{n\pi t}{2}) \cdot (\frac{2}{n\pi})^2 \right]_{0}^{1} \right\} , \\
=& \frac{1}{2}\left[-\frac{2}{n\pi} + \frac{4}{n^2\pi^2} sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2} sin\frac{n\pi}{2} + \frac{2}{n\pi} \right] , \\
=& 0 ,
\end{aligned}
$$

Substituting the values of $a's$ and $b's$ in (1), we get

$$
f(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \left(1 - cos\frac{n\pi}{2} \right) cos\frac{n\pi t}{2} .
$$

Ex. 7

Expand $f(x)$ as the Fourier series of sine terms.

$$f(x) = \frac{1}{4} - x \ , \ \text{if } 0 < x < \frac{1}{2}$$
$$= x - \frac{3}{4} \ , \ \text{if } \frac{1}{2} < x < 1.$$

Solution: Let

$$f(x) = \sum_{n=1}^{\infty} b_n sin \frac{n\pi x}{c} = \sum_{n=1}^{\infty} b_n sin \frac{n\pi x}{1} \qquad (1)$$

where

$$\begin{aligned} b_n =& \frac{2}{c} \int_0^c f(x) sin \frac{n\pi x}{c} \\ =& \frac{2}{1} \Big[\int_0^{1/2} (\frac{1}{4} - x) sin \frac{n\pi x}{1} dx + \int_{1/2}^{1} (x - \frac{3}{4}) sin \frac{n\pi x}{1} dx \Big] \end{aligned}$$

$$
\begin{aligned}
b_n =& 2\left[(\frac{1}{4} - x)(-\frac{cosn\pi x}{n\pi}) - (-1)(-\frac{sinn\pi x}{n^2\pi^2}) \right]_0^{1/2} + \\
&+ 2\left[(x - \frac{3}{4})(-\frac{cosn\pi x}{n\pi}) - (1)(-\frac{sinn\pi x}{n^2\pi^2}) \right]_{1/2}^{1} \\
=& 2\left[\frac{1}{4n\pi} cos\frac{n\pi}{2} - \frac{1}{n^2\pi^2} sin\frac{n\pi}{2} + \frac{1}{4n\pi} \right] + \\
&+ 2\left[-\frac{1}{4n\pi} cosn\pi + 0 - \frac{1}{4n\pi} cos\frac{n\pi}{2} - \frac{1}{n^2\pi^2} sin\frac{n\pi}{2} \right] \\
=& \frac{1}{2n\pi}[1 - (-1)^n] - 4\frac{sinn\pi/2}{n^2\pi^2} \ .
\end{aligned}
$$

Hence

$$
f(x) = (\frac{1}{\pi} - \frac{4}{\pi^2}) sin\pi x + (\frac{1}{3\pi} - \frac{4}{3^2\pi^2}) sin3\pi x + (\frac{1}{5\pi} - \frac{4}{5^2\pi^2}) sin5\pi x + ...
$$

Ex. 8

Obtain a half range cosine series for

$$
\begin{aligned}
f(x) &= kx \text{ , } 0 \leq x \leq l/2 \\
&= k(l-x) \text{ , } l/2 \leq x \leq l.
\end{aligned}
$$

Deduce the sum of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\infty$$

Solution: Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos \frac{n\pi x}{l} \tag{1}$$

where

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\left[\int_0^{l/2} kxdx + \int_{l/2}^{l} k(l-x)dx\right] = \frac{kl}{2}$$

$$
\begin{aligned}
a_n =& \frac{2}{l}\left[\int_0^{l/2} kx cos\frac{n\pi x}{l}dx + \int_{l/2}^{l} k(l-x)cos\frac{n\pi x}{l}dx\right] \\
=& \frac{2k}{l}\left[x(sin\frac{n\pi x}{l})\cdot(\frac{l}{n\pi}) - (1)(-cos\frac{n\pi x}{l})\cdot(\frac{l^2}{n^2\pi^2})\right]_0^{l/2} + \\
&+ \frac{2k}{l}\left[(l-x)(sin\frac{n\pi x}{l})\cdot(\frac{l}{n\pi}) - (-1)(-cos\frac{n\pi x}{l})\cdot(\frac{l^2}{n^2\pi^2})\right]_{l/2}^{l} \\
=& \frac{2k}{l}\left[\frac{l^2}{2n\pi}sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\right] + \\
&+ \frac{2k}{l}\left[-\frac{l^2}{n^2\pi^2}cos n\pi - \frac{l^2}{2n\pi}sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}cos\frac{n\pi}{2}\right] \\
=& \frac{2kl}{n^2\pi^2}[2cos\frac{n\pi}{2} - 1 - (-1)^n]
\end{aligned}
$$

Substituting values of a_0 and a_n in series (1), we get

$$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}cos\frac{2\pi x}{l} + \frac{1}{6^2}cos\frac{6\pi x}{l} + \frac{1}{10^2}cos\frac{10\pi x}{l} + ...\right]$$

Putting $x = l$, we get

$$0 = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + ...\infty\right]$$

On simplifying, we get

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\infty = \frac{\pi^2}{8}$$

Some useful results:

$$2sinAcosB = sin(A+B) + sin(A-B)\ , 2sin^2x = 1 - cos2x$$
$$2cosAcosB = cos(A+B) + cos(A-B)\ , 2cos^2x = 1 + cos2x$$
$$2sinAsinB = cos(A-B) - cos(A+B)$$

THANK YOU