B. Sc. I Semester II

Paper No. III

DIFFERENTIAL EQUATIONS

Chapter Scheme

1. Differential Equations of First order and first degree
2. Linear Differential Equations With Constant Coefficients
3. Homogeneous Linear Differential Equations
4. Differential equations of first order but degree higher than first

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- **Definition:** An equation involving differential coefficient or differentials is called differential equation.
- If the differential coefficient be ordinary the equation is called ordinary differential equation.
- If they be partial differential coefficient is called partial differential equation.

1. $\frac{dy}{dx} - cosx = 0$

2. $\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{1}{x}y = 1$

3. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$

4. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 16y = 0$

Above all equations are differential equations out of which 3rd equation is partial differential equation and other are ordinary differential equations.

Order &degree of a differential equation:

- The order of a differential equation is the order of the highest order derivative involved in the differential equation.
- The degree of a differential equation is the power of the highest derivative involved in the equation.

Determine order and degree of the following equations

$$y = x\frac{dy}{dx} + \frac{2}{\frac{dy}{dx}}$$

- order=1 degree = 2

$$y = x\frac{dy}{dx} + 4\sqrt{1 + \left(\frac{dy}{dx}\right)^3}$$

- order =1 degree = 3

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4x = \sqrt[3]{1 + \frac{d^3y}{dx^3}}$$

- order = 3 degree = 1

- **Solution of differential equation:**
- A relation in all the variables, dependent as well as independent which does not contain any derivative and satisfies the given differential equation, is called a solution of that differential equation.
- **General Solution of differential equation:**
- A solution of differential equation along with all its arbitrary constants, is called general Solution of differential equation.
- **Particular Solution of differential equation:**
- By assuming specific values to the arbitrary constants, as per the initial conditions given along with the differential equation, we get a particular solution of differential equation.

The ordinary differential equation of first order and first degree

$$\frac{dy}{dx} = f(x, y)$$

- The various methods available to solve above equation can be classified as

- Method of separation of variables
- Equations homogenous in x and y
- Non-homogenous equation
- Exact differential equation
- Linear differential equation

Type 1: Method of separation of variables

General form

$$f(x,y)dx + g(x,y)dy = 0$$

Where $f(x,y)$ and $g(x,y)$ are functions of x & y

If after rearrangement, the above equation can be written as

$$f(x)dx + g(y)dy = 0$$

i.e. variable separable form

Its solution, obtained by integration, given by

$$\int f(x)dx + \int g(y)dy = c$$

Where c is a constant of integration

Ex 1: Solve

$$(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$$

Solution: Dividing by $(e^y + 1)\sin x$

$$\frac{\cos x}{\sin x}dx + \frac{e^y}{e^y + 1}dy = 0$$

i.e in variable separable form

Integrating we get

$$\log \sin x + \log(e^y + 1) = c$$

As required solution

Ex 2: Solve

$$(2x-2y+3)dx-(x-y+1)dy=0$$

Solution: Given equation can be written as

$$\frac{dy}{dx}=\frac{2x-2y+3}{x-y+1}$$

$$=\frac{2(x-y)+3}{(x-y)+1}\ldots\ldots\ldots(1)$$

Put $(x-y)=v$

$$\therefore 1-\frac{dy}{dx}=\frac{dv}{dx}$$

$$\therefore \frac{dy}{dx}=1-\frac{dv}{dx}$$

Hence equation (1) becomes

$$1-\frac{dv}{dx}=\frac{2v+3}{v+1}$$

$$\frac{dv}{dx}=1-\frac{2v+3}{v+1}$$

$$=\frac{-v-2}{v+1}$$

$$=-\frac{v+2}{v+1}$$

$$\frac{v+1}{v+2}\,dv=-dx$$

Integrating both sides we get

$$\int\frac{v+1}{v+2}\,dv=-\int dx+c$$

$$\int[1-1/(v+2)]dv=-x+c$$

$$\int dv-\int\frac{1}{v+2}dv=-x+c$$

$$v-\log(v+2)=-x+c$$

$$x-y-\log(x-y+2)=-x+c$$

$$2x-y-\log(x-y+2)=c$$

Where c is constant of integration.
Which is required solution.

Homogenous Equations:

An equation of the type $$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

Where f and g are homogenous expressions in x and y and of same is called homogenous differential of 1st order & 1st degree.

Method of solution:

Given equation is $$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \dots\dots\dots\dots (1)$$

Put y =v x

Differentiate w. r. t. x $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Hence equation (1) becomes $$\frac{dv}{F(v) - v} = \frac{dx}{x}$$

Which is in variable separable type. Now integrating & resubstuting v = y/x we get required solution.

Ex 1: Solve $x^2ydx-(x^3+y^3)dy=0$

Solution: Given equation can be written as

$$\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$$

Put y =v x

Differentiate w. r. t. x

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Hence equation (1) becomes

$$\therefore v+\frac{xdv}{dx}=\frac{x^3v}{x^3+x^3v^3}$$

$$\therefore \frac{xdv}{dx}=-\frac{v^4}{1+v^3}$$

$$\therefore \frac{1+v^3}{v^4}=-\frac{dx}{x}$$

integrating we get

$$logy-\frac{x^3}{3y^3}=c$$

Which is required solution.

Ex 2: Solve

$$y^2+x^2\frac{dy}{dx}=xy\frac{dy}{dx}$$

Non-homogenous equation:

An equation of the type $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$

Where a, b, c, a’, b’, c’ are constants is called Non-homogenous differential equation of 1^{st} order & 1^{st} degree.

Method of solution:

Given equation is

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \ldots\ldots\ldots..(1)$$

When $\frac{a}{a'} \neq \frac{b}{b'}$

Use the transformation equations

$$x = X + h, \quad y = Y + k$$

$$\therefore dx = dX, \quad dy = dY$$

Hence we have

$$ax + by + c = aX + bY + (ah + bk + c)$$

And

$$a'x + b'y + c' = a'X + b'Y + (a'h + b'k + c')$$

Now find the values of h & k by solving simultaneously

$$ah + bk + c = 0 \quad \& \quad a'h + b'k + c' = 0$$

Let these values be $h = \frac{bc' - cb'}{ab' - a'b}$, $k = \frac{ca' - c'a}{ab' - a'b}$

Provided that $ab' - a'b \neq 0$

Using above values in equation (1) we get

$$\frac{dy}{dx} = \frac{aX+bY}{a'X+b'Y}$$

This is homogenous equation & can be solved by the method of homogenous equation.

If the solution of this equation be, F(X,Y) = C then solution of the equation (1) is F(x-h , y-k) = c

Ex 1: Solve $(3y - 7x + 7)dx + (7y - 3x + 3)dy = 0$

Ex 2: Solve $(2x - y + 1)dx + (2y - x - 1)dy = 0$

Solution: Given equation can be written as

$$\frac{dy}{dx} = \frac{y - 2x - 1}{2y - x - 1} \ldots\ldots\ldots\ldots\ldots (1)$$

Here a =- 2 , b = 1 , a' = -1 , b' = 2

$$\therefore \frac{a}{a'} \neq \frac{b}{b'}$$

Use the transformation equations

$$x = X + h, \quad y = Y + k$$

$$\therefore dx = dX, \quad dy = dY$$

Hence we have

$$y - 2x - 1 = -2X + Y + (-2h + k - 1)$$

And

$$2y - x - 1 = -X + 2Y + (-h + 2k - 1)$$

Now find the values of h & k by solving simultaneously

$$-2h + k - 1 = 0 \quad \& \quad -h + 2k - 1 = 0$$

$$\therefore h = -\frac{1}{3} \quad , \quad k = \frac{1}{3}$$

Using above values in equation (1) we get

$$\frac{dy}{dx} = \frac{-2X + Y}{-X + 2Y} = \frac{2X - Y}{2Y - X} \ldots\ldots\ldots (2)$$

This is homogenous equation.

$$put\ Y = vX$$

$$\therefore dx = dX \quad , \quad dy = dY$$

Hence equation (2) becomes

$$v + X\frac{dv}{dX} = \frac{2X - vX}{2vX - X}$$

Simplifying and integrating we get

$$(X - Y)^{\frac{1}{2}}(X + Y)^{-\frac{3}{2}}X^{-1} = c$$

Now putting X= x+1/3 Y= y-1/3 we get required solution.

Exact differential equation:

A differential equation which is obtained from its primitive by differentiation only and without any operation of elimination or reduction is called as exact differential equation.

If u = c be the primitive where u is a function of x, y then on differentiating we get

$$du = 0 \quad i.e. \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

This is exact differential equation.

Necessary & sufficient condition for exactness:

The necessary & sufficient condition for the equation M dx + N dy = 0 be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Method of solution:

$$\int_{y=constant} M\, dx + \int (terms\ in\ N\ free\ from\ x\ )\, dy = c$$

Theorem: To determine the necessary and sufficient condition for a differential equation of first order and first degree to be exact

Statement: The necessary and sufficient condition for the differential equation $M\,dx + N\,dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Proof: Let $M\,dx + N\,dy = 0$ .... (1) be exact, to prove $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ....(2)

Necessary Condition: Let (1) be exact. Hence by definition, there exist a function $f(x, y)$ of x and y, such that $d[f(x, y)] = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$.

..........(3)

Equating coefficients of dx and dy in (3), we get

$$M = \frac{\partial f}{\partial x} \quad \text{.....................} \ (4)$$

and

$$N = \frac{\partial f}{\partial y} \quad \text{.....................} \ (5)$$

To remove the unknown function $f(x,y)$, we differentiate partially (4) and (5) with respect to y and x respectively giving

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(6)}$$

And $$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(7)}$$

Since $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, (6) and (7) give $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

The condition is sufficient: **we assume that (2) holds and show that (1) is an exact equation. For this we must find a function $f(x,y)$, such that $d[f(x,y)] = Mdx + Ndy$.**

Let $g(x,y) = \int M\,dx$ .... (8)

Be a partial integral of M, that is, the integral obtained by keeping y fixed. We first prove that $\left(N - \frac{\partial g}{\partial y}\right)$ is a function of y only. This is clear because

$$\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x \partial y} = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y \partial x} \quad \text{as} \quad \frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \quad \text{, using (8)}$$

$$= 0 \quad \text{, using (2)}$$

Take $f(x,y) = g(x,y) + \int \left(N - \frac{\partial g}{\partial y}\right) dy$ .... (9)

Hence on total differentiation of (9), we get

$$df = dg + \left(N - \frac{\partial g}{\partial y}\right) dy = \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy\right) + N dy - \frac{\partial g}{\partial y} dy$$

$$= \left(\frac{\partial g}{\partial x}\right) dx + N dy = M dx + N dy \quad \text{, using (8)}$$

Thus, if (2) is satisfied, (1) is surely an exact equation.

Ex 1: Solve

$$\left[y\left(1+\frac{1}{x}\right)+cosy\right]dx+[x+logx-xsiny]dy=0$$

Solution: Here

$$M=y\left(1+\frac{1}{x}\right)+cosy \;,\; N=x+logx-xsiny$$

$$\therefore \frac{\partial M}{\partial y}=1+\frac{1}{x}-siny \;,\; \frac{\partial N}{\partial x}=1+\frac{1}{x}-siny$$

$$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow Given\ equation\ is\ exact$$

$\therefore$ *Solution is*

$$\int_{y=constant} M\,dx+\int (terms\ in\ N\ free\ from\ x\ )\,dy=c$$

$$\therefore \int_{y=constant} y\left(1+\frac{1}{x}\right)+cosy\,dx+\int 0\,dy=c$$

$$y[x+logx]+xcosy=c$$

Ex 2: Solve

$$(x^2-ay)dx+(y^2-ax)dy=0$$

Solution: Here

$$M=(x^2-ay)\;,\;N=(y^2-ax)$$

$$\therefore \frac{\partial M}{\partial y}=-a \;,\; \frac{\partial N}{\partial x}=-a$$

$$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow Given\ equation\ is\ exact$$

$\therefore$ *Solution is*

$$\int_{y=constant} M\,dx+\int (terms\ in\ N\ free\ from\ x\ )\,dy=c$$

$$\therefore \int_{y=constant} (x^2-ay)\,dx+\int y^2\,dy=c$$

$$\therefore \frac{x^3}{3}-axy+\frac{y^3}{3}=c$$

Equations reducible to exact differential equation:

Sometimes the equation M dx + N dy = 0 is not exact differential equation. To reduce this equation to exact form we multiply the equation by a suitable factor is called as integrating factor.

Rules for finding I.F.:-

- If the equation $Mdx + Ndy = 0$ *is homogenous equation and* $Mx + Ny \neq 0$ *then*

$$I.F. = \frac{1}{Mx + Ny}$$

- If the equation $Mdx + Ndy = 0$ *is of the form* $f_1(xy)ydx + f_2(xy)xdy = 0$ *and* $Mx - Ny \neq 0$ *then* $I.F. = \frac{1}{Mx - Ny}$

- If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ *is a function of x or constant only say* $f(x)$ *then* $I.F. = e^{\int f(x)\,dx}$

- If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ *is function of y or constant only say* $f(y)$ *then* $I.F. = e^{\int f(y)dy}$

Ex 1: Solve $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

Solution: Here

$$M=(x^2y-2xy^2)\ ,\ N=-(x^3-3x^2y)$$

$$\therefore \frac{\partial M}{\partial y}=x^2-4xy \quad , \quad \frac{\partial N}{\partial x}=6xy-3x^2$$

$$\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \Rightarrow \text{Given equation is non exact}$$

Since given equation is homogenous, hence $I.F.=\frac{1}{Mx+Ny}=\frac{1}{x^2y^2}$

Now multiplying given equation by I. F. we get exact differential equation as

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(\frac{3}{y}-\frac{x}{y^2}\right)dy=0$$

$$\therefore \text{Solution is} \int_{y=constant} M\,dx+\int(\text{terms in } N \text{ free from } x)\,dy=c$$

$$\therefore \int_{y=constant}\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

$$\therefore \frac{x}{y}-2\log x+3\log y=c$$

Ex 2: Solve $y(xy+2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Solution: Here

$$M = y(xy+2x^2y^2)\ ,\ N = x(xy - x^2y^2)$$

$$\therefore \frac{\partial M}{\partial y} = 2xy + 6x^2y^2 \quad , \frac{\partial N}{\partial x} = 2xy - 3x^2y^2$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow Given\ equation\ is\ non\ exact$$

Since given equation is of the form $f_1(xy)ydx + f_2(xy)xdy = 0 \quad \therefore I.F. = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$

Now multiplying given equation by I. F. we get exact differential equation as

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

$$\therefore Solution\ is \int_{y=constant} M\,dx + \int (terms\ in\ N\ free\ from\ x\ )\,dy = c$$

$$\therefore \int_{y=constant} \left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \int -\frac{1}{y}dy = c$$

$$\therefore 2\log\left(\frac{x}{y}\right) - \frac{1}{xy} = c$$

Ex 3: Solve $(x^2+y^2+2x)dx+2ydy=0$

Solution: Here $M=x^2+y^2+2x\ ,\ N=2y$

$$\therefore \frac{\partial M}{\partial y}=2y \quad , \quad \frac{\partial N}{\partial x}=0$$

$$\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \Rightarrow \textit{Given equation is non exact}$$

Now $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\dfrac{2y-0}{2y}=1=f(x)say$

$$\therefore I.F.=e^{\int f(x)\,dx}=e^{\int 1dx}=e^x$$

Now multiplying given equation by I. F. we get exact differential equation as

$$e^x(x^2+y^2+2x)dx+e^x 2ydy=0$$

$$\therefore \textit{Solution is} \int_{y=constant} M\,dx+\int(\textit{terms in } N \textit{ free from } x\,)\,dy=c$$

$$\therefore \int_{y=constant}\left(e^x(x^2+y^2+2x)\right)dx+\int 0\,dy=c$$

$$\therefore (x^2+y^2)e^x=c$$

Ex 4: Solve $(3x^2y^4+2xy)dx+(2x^3y^3-x^2)dy=0$

Solution: Here

$$M=3x^2y^4+2xy\,,\;N=2x^3y^3-x^2$$

$$\therefore \frac{\partial M}{\partial y}=12x^2y^3+2x \quad , \quad \frac{\partial N}{\partial x}=6x^3y^2-2x$$

$$\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \Rightarrow \text{Given equation is non exact}$$

Now $$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{-2}{y}=f(y)say$$

$$\therefore I.F.=e^{\int f(y)dx}=e^{\int -\frac{2}{y}dy}=e^{-2logy}=y^{-2}$$

Now multiplying given equation by I. F. we get exact differential equation as

$$y^{-2}(3x^2y^4+2xy)dx+y^{-2}(2x^3y^3-x^2)dy=0$$

$$\therefore \text{Solution is} \int_{y=constant} M\,dx+\int(\text{terms in } N \text{ free from } x\,)\,dy=c$$

$$\therefore \int_{y=constant}(3x^2y^2+\frac{2x}{y})dx+\int 0\,dy=c$$

$$\therefore x^3y^2+\frac{x^2}{y}=c$$

Linear differential equation:

A differential equation is said to be linear if the dependant variable and its derivative appears in the equation in 1st degree only.
The general form of differential equation of 1st order & 1st degree is

$$\frac{dy}{dx} + Py = Q$$

Where P & Q are functions of x or constant only.

Method of solution:

First write the equation with coefficient of dy/dx as unity, now find

$$I.F. = e^{\int P dx} \text{ and further solution as}$$

$$y.I.F = \int I.F.Q dx + c$$

Ex 1: solve $\cos^2 x \frac{dy}{dx} + y = \tan x$

Solution: Dividing by $\cos^2 x$

$$\frac{dy}{dx} + \sec^2 x \,.\, y = \tan x \sec^2 x$$

$$\therefore P = \sec^2 x \;, Q = \tan x \sec^2 x$$

$$I.F. = e^{\int P dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

Hence solution is,

$$\therefore y.I.F. = \int I.F.Q dx + c$$

$$\therefore y e^{\tan x} = \int e^{\tan x} \tan x \sec^2 x dx + c$$

$$\therefore y e^{\tan x} = e^{\tan x}(\tan x - 1) + c$$

Ex 2: Solve

$$\frac{dy}{dx} + \frac{2x}{x^2+1} y = \frac{4x^2}{x^2+1}$$

Ex 3: Solve

$$\frac{dy}{dx} - 2y\cos x = -2\sin 2x$$

Bernoulli's equation:

An equation of the type

$$\frac{dy}{dx} + Py = Qy^n$$

Where P & Q are functions of x or constant only is called Bernoulli's equation.

It can be reduced to linear form by putting $y^{n-1} = z$

Ex 1: Solve

$$\frac{dy}{dx} = x^3y^3 - xy$$

Solution: Given equation can be written as

$$\frac{dy}{dx} + xy = x^3y^3$$

Dividing by y^3 and putting $y^{-2} = z$, we get

$$\frac{dy}{dz} - 2xz = -2x^3$$

This is linear equation and its solution is given by

$$\therefore y^{-2}e^{-x^2} = e^{-x^2}(x^2+1) + c$$

Ex 2: Solve $tany\frac{dy}{dx} + tanx = cosycos^3x$

Ex 3: Solve $x\frac{dy}{dx} + y = x^2y^2$

Linear differential equations with constant coefficients

Definition: An equation of the type

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots \ldots \ldots \ldots . + a_n \frac{d^n y}{dx^n} = X$$

Where $a_1, a_2, \ldots \ldots \ldots . a_n$ are constants and X is a function of x only is called as linear differential equation with constant coefficients.

If d/dx =D then above equation can be written as f(D)y = X

The equation f(D)y = 0 is called as A.E.

The general solution is : y = C.F. + P.I.

Methods to find C.F. :

- If roots of auxiliary equations are real &distinct then

$$c.f. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots$$

- If roots of auxiliary equations are real &repeated then

$$c.f. = (c_1 + c_2 x + c_3 x^2)e^{m_1 x} + \cdots$$

Where the root m_1 repeated 3 times

- If roots of auxiliary equations are imaginary &distinct then

$$c.f. = e^{\alpha x}(c_1 cos\beta x + c_2 sin\beta x)$$

Where α+iβ & α-iβ are imaginary roots.

- If roots of auxiliary equations are imaginary & repeated then

$$c.f. = e^{\alpha x}\{(c_1 + c_2 x)cos\beta x + (c_3 + c_4 x)sin\beta x)\}$$

Where α+iβ & α-iβ are repeated imaginary roots.

Case 1: Solution of f(D)y = 0

Ex 1: Solve $(D^3 + 7D^2 + 16D + 10)y = 0$

Ex 2: Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Ex 3: Solve $\frac{d^4y}{dx^4} + 4y = 0$

Ex 4: Solve $(D^4 + 2D^2 + 1)y = 0$

Ex 5: Solve $\frac{d^6y}{dx^6} + 64y = 0$

Inverse operator $\frac{1}{f(D)}$ & symbolical expression for P.I.-

We define $$\frac{1}{f(D)}X$$

as that function of x not containing arbitrary constants which when acted upon by f(D) gives X.

Hence by definition

$$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

And hence $$\frac{1}{f(D)}X$$

satisfies the equation f(D)y = X and so is a P.I. of the equation f(D)y = X. Thus symbolically P.I. is given by

$$P.I. = \frac{1}{f(D)}X$$

Inverse operators:

$$\frac{1}{D} X = \int X \, dx$$

$$\frac{1}{D-a} X = e^{ax} \int e^{-ax} X \, dx$$

$$\frac{1}{D+a} X = e^{-ax} \int e^{ax} X \, dx$$

If f(a)=0 then

$$P.I. = \frac{x}{f'(a)} e^{ax} \quad , f'(a) \neq 0$$

$$if \; f'(a) = 0$$

$$P.I. = \frac{x^2}{f''(a)} e^{ax} \quad , f''(a) \neq 0 \quad and\ so\ on$$

Procedure to find P.I. :

Case 1 : when $X = e^{ax}$

$$\text{P.I.} = \frac{1}{F(D)} e^{ax}$$

$$= \frac{1}{F(a)} e^{ax}$$

where f(a) $\neq$ 0

Ex 1: Solve $\frac{d^3y}{dx^3} + y = 3e^x + 5$

Ex 2: Solve $(D^3 - 3D^2 + 4)y = e^{2x}$

Solution: A.E.is

$$D^3 + 1 = 0$$

Solving we get roots as

$$D = -1 \;, D = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\therefore C.F. = c_1 e^{-1x} + e^{\frac{1}{2}x}\left(c_2 cos\left(\frac{\sqrt{3}}{2}\right)x + c_3 sin\left(\frac{\sqrt{3}}{2}\right)x\right.$$

$$P.I. = \frac{1}{D^3 + 1}(3e^x + 5)$$

$$= 3\frac{1}{D^3+1}e^x + 5\frac{1}{D^3+1}e^{0x}$$

$$= 3\frac{1}{1+1}e^x + 5\frac{1}{0+1}e^{0x}$$

$$= \frac{3}{2}e^x + 5$$

$$\therefore solution\ is\ y = C.F. + P.I.$$

Solution: A.E.is

$$D^3 - 3D^2 + 4 = 0$$

Solving we get roots as

$$D = -1\,, 2\,, 2$$

$$\therefore C.F. = c_1 e^{-1x} + (c_2 + c_3 x)e^{2x}$$

$$P.I. = \frac{1}{D^3 - 3D^2 + 4}e^{2x}$$

$$= \frac{x}{3D^2 - 6D}e^{2x}$$

$$= x\frac{x}{6D-6}e^{2x}$$

$$= \frac{x^2}{6}e^{2x}$$

$$\therefore solution\ is\ y = C.F. + P.I.$$

Case2: When $X = \sin ax$ or $\cos ax$

$$P.I. = \frac{1}{\emptyset(D^2)} Sinax$$

$$= \frac{1}{\emptyset(-a^2)} \sin ax \quad \text{where } \emptyset(-a^2) \neq 0$$

When $\emptyset(-a^2) = 0$ then

$$P.I. = \frac{x}{\emptyset'(-a^2)} \sin ax \;, \emptyset'(-a^2) \neq 0$$

if $\emptyset'(-a^2) = 0$ then

$$P.I = \frac{x^2}{\emptyset''(-a^2)} \sin ax \quad , \emptyset''(-a^2) \neq 0 \quad \text{and so on}$$

Ex 1: Solve $(D^2 - 2D + 5)y = \sin 3x$

Ex 2: Solve $(D^3 - 3D^2 + 9D - 27)y = \cos 3x$

Solution: A.E.is

$$D^2 - 2D + 5 = 0$$

Solving we get roots as

$$D = 1 \pm 2i$$

$$\therefore C.F. = e^{-x}(c_1 cos2x + c_2 sin2x)$$

$$P.I. = \frac{1}{D^2 - 2D + 5} sin3x$$

$$= \frac{1}{-9-2D+5} sin3x$$

$$= -\frac{1}{2}\left(\frac{1}{D+2}\right) sin3x$$

Rationalizing denominator we get

$$= -\frac{1}{2}\left(\frac{D-2}{D^2-4}\right) sin3x$$

$$= -\frac{1}{2}\left(\frac{D-2}{-9-4}\right) sin3x$$

$$= \frac{1}{26}(3cos3x - 2sin3x$$

$$\therefore solution\ is\ y = C.F. + P.I.$$

Ex 3: Solve $\frac{d^2y}{dx^2} + 9y = cos3x$

Solution: A.E.is

$$D^2 + 9 = 0$$

Solving we get roots as $D = \pm 3i$

$$\therefore C.F. = (c_1 cos3x + c_2 sin3x)$$

$$P.I. = \frac{1}{D^2 + 9} cos3x$$

$$= \frac{x}{2D} cos3x$$

$$= \frac{x}{2}\int cos3x\ dx$$

$$= \frac{x}{6} sin3x$$

$$\therefore solution\ is\ y = C.F. + P.I.$$

Case 3 : when X= x^m where m is a positive integer

$$P.I. = \frac{1}{f(D)}\, x^m$$

$$= \{1 + \emptyset(D)\}^{-1}\, x^m$$

$$\text{Where } (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

Ex 1 Solve $(D^2 - 3D + 2)y = x$

Ex 2: Solve $(D^3 + 2D^2 + D)y = x^2 + x$

Solution: A.E.is

$$(D^2 - 3D + 2) = 0$$

Solving we get roots as

$$D = 1\,,2$$

$$\therefore C.F. = c_1 e^{x} + c_2 e^{2x}$$

$$P.I. = \frac{1}{D^2 - 3D + 2} x$$

$$= \frac{1}{2}\left(1/\left\{1 + \frac{D^2 - 3D}{2}\right\}\right) x$$

$$= \frac{1}{2}\left\{1 + \frac{D^2 - 3D}{2}\right\}^{-1} x$$

$$= \frac{1}{2}\left\{1 - \frac{D^2 - 3D}{2} + \frac{D^2 - 3D}{2} - \cdots \ldots \ldots\right\} x$$

$$= \frac{1}{2}\left\{x + \frac{3}{2}\right\}$$

$$\therefore solution\ is\ y = C.F. + P.I.$$

Solution: A.E.is

$$D^3 + 2D^2 + D = 0$$

Solving we get roots as

$$D = 0\,, -1\,, -1$$

$$\therefore C.F. = c_1 e^{0x} + (c_2 + c_3 x) e^{-x}$$

$$P.I. = \frac{1}{D^3 + 2D^2 + D}(x^2 + x)$$

$$= \frac{1}{D\{1 + (D^2 + 2D\}} e^{2x}$$

$$= \frac{1}{D}\{1 + (D^2 + 2D)\}^{-1}(x^2 + x)$$

$$= \frac{1}{D}\{1 - (D^2 + 2D) + (D^2 + 2D)^2 - \cdots \ldots\}(x^2 + x)$$

$$= \frac{1}{D}\{x^2 + x - 2 - 4x - 2 + 8\}$$

$$= \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

$$\therefore solution\ is\ y = C.F. + P.I.$$

Case 4 : when $X = e^{ax}v$ where V is a function of x

$$P.I. = \frac{1}{f(D)} e^{ax} V$$

$$= e^{ax} \frac{1}{f(D+a)} V$$

And use case 2 or case3

Ex 1: Solve $(D^2 + 2)y = e^x cosx$

Ex 2: Solve $(D^2 + 3D + 2)y = e^{3x} x$

Solution: A.E.is

$$(D^2+2)=0$$

Solving we get roots as

$$D=\pm\sqrt{2}\,i$$

$$\therefore C.F.=c_1\cos\sqrt{2}\,x+c_2\sin\sqrt{2}\,x$$

$$P.I.=\frac{1}{D^2+2}e^x\cos x$$

$$=e^x\frac{1}{(D+1)^2+2}\cos x$$

$$=e^x\frac{1}{D^2+2D+3}\cos x$$

$$=e^x\frac{1}{-1+2D+3}\cos x$$

$$=\frac{e^x}{2}\left\{\frac{1}{D+1}\right\}\cos x$$

$$=\frac{e^x}{2}\left\{\frac{D-1}{D^2-1}\right\}\cos x$$

$$=-\frac{e^x}{4}(\cos x+\sin x)$$

$$\therefore solution\ is\ y=C.F.+P.I.$$

Solution: A.E.is

$$(D^2+3D+2)=0$$

Solving we get roots as

$$D=-1,-2$$

$$\therefore C.F.=c_1e^{-x}+c_2e^{-2x}$$

$$P.I.=\frac{1}{D^2+3D+2}e^{3x}x$$

$$=e^{3x}\frac{1}{(D+3)^2+3(D+2)+2}x$$

$$=e^{3x}\frac{1}{D^2+9D+20}x$$

$$=\frac{e^{3x}}{20}\left\{\frac{1}{1+\frac{D^2+9D}{20}}\right\}x$$

$$=\frac{e^{3x}}{20}\left\{1+\frac{D^2+9D}{20}\right\}^{-1}x$$

$$=\frac{e^{3x}}{20}\left\{1-\frac{D^2+9D}{20}+\frac{D^2+9D}{20}-\cdots\right\}x$$

$$=\frac{e^{3x}}{20}\left(x-\frac{9}{20}\right)$$

Case 5 : *When $X = xV$*

$$P.I. = \frac{1}{f(D)} xV$$

$$= \left\{x - \frac{f'(D)}{f(D)}\right\} \frac{1}{f(D)} V$$

Ex 1: Solve $(D^2 + 4)y = x\sin x$

Ex 2: Solve $(D^2 + 2D + 1)y = x\cos x$

Solution: A.E.is

$$D^2 + 4 = 0$$

Solving we get roots as

$$D = \pm 2i$$

$$\therefore C.F. = (c_1 cos2x + c_2 sin2x)$$

$$P.I. = \frac{1}{D^2+4} xsinx$$

$$= \left\{x - \frac{2D}{D^2+4}\right\} \frac{1}{D^2+4} \, sinx$$

$$= \left\{x - \frac{2D}{D^2+4}\right\} \frac{1}{(-1+4)} \, sinx$$

$$= \frac{1}{3}\left\{x - \frac{2D}{D^2+4}\right\} sinx$$

$$= \frac{1}{3}\left\{xsinx - \frac{2D}{D^2+4} sinx\right\}$$

$$= \frac{1}{3}\left\{xsinx - \frac{2D}{-1+4} sinx\right\}$$

$$= \frac{1}{3}\left\{xsinx - \frac{2}{3} cosx\right\}$$

$$\therefore solution \ is \ y = C.F. + P.I.$$

Solution: A.E.is

$$D^2 + 2D + 1 = 0$$

Solving we get roots as

$$D = -1, -1$$

$$\therefore C.F. = (c_1 + c_2 x)e^{-x}$$

$$P.I. = \frac{1}{D^2 + 2D + 1} x\cos x$$

$$= \left\{x - \frac{2D}{D^2 + 2D + 1}\right\} \frac{1}{D^2 + 2D + 1} \cos x$$

$$= \left\{x - \frac{2D + 2}{D^2 + 2D + 1}\right\} \frac{1}{(-1 + 2D + 1)} \cos x$$

$$= \frac{1}{2}\left\{x - \frac{2D + 2}{D^2 + 2D + 1}\right\} \frac{1}{D} \cos x$$

$$= \frac{1}{2}\left\{x - \frac{2D + 2}{D^2 + 2D + 1}\right\} \int \cos x dx$$

$$= \frac{1}{2}\left\{x - \frac{2D + 2}{D^2 + 2D + 1}\right\} \sin x$$

$$= \frac{1}{2}\left\{x\sin x - \frac{2D + 2}{-1 + 2D + 1} \sin x\right\}$$

$$= \frac{1}{2}\left\{x\sin x - (1 + \frac{1}{D}) \sin x\right\}$$

$$= \frac{1}{2}\left\{x\sin x - \sin x - \int \sin x dx\right\}$$

$$= \frac{1}{2}\{x\sin x - \sin x + \cos x\}$$

$$\therefore \text{solution is } y = C.F. + P.I$$

Homogenous linear Differential equations with constant Coefficients

Definition: An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = X \text{ ....(1)}$$

Where $a_1, a_2, a_3, \ldots . a_n$ *are constants* & X *is a function of x only* is called as Homogenous linear differential equation with constant coefficients.

Method of Solution: First we can change the independent variable x to z by using the substitution

$$\log x = z \text{ i.e. } x = e^z$$

&

$$x \frac{dy}{dx} = Dy \quad (Since\ D = \frac{d}{dz}$$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

....

....

$$x^n \frac{d^n y}{dx^n} = D(D-1)(D-2) \ldots . (D-n+1)y$$

Using above values in equation (1) we get

$$\frac{d^n y}{dz^n} + a_1 \frac{d^{n-1} y}{dz^{n-1}} + a_2 \frac{d^{n-2} y}{dz^{n-2}} + \cdots + a_n y = Z$$

Which is linear differential with constant coefficients and can be solved by using methods of L. D.

Examples:

1) $\boldsymbol{x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x}$

2) $\boldsymbol{(x^3D^3 + x^2D^2 - 2)y = x - \frac{1}{x^3}}$

3) $\boldsymbol{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = sin(logx^2)}$

4) $\boldsymbol{x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = xlogx}$

Legendre's Linear Differential Equation with constant coefficients:

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 (a+bx)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = X \text{ ....(1)}$$

Where $a_1, a_2, a_3, \ldots . a_n$ *are constants* & X *is a function of x only* is called as Legendre's linear differential equation with constant coefficients.

Method of Solution: First we can change the independent variable x to z by using the substitution

$$\log(a+bx) = z \quad i.e. a+bx = e^z$$

& $$(a+bx)\frac{dy}{dx} = bDy \qquad (Since\ D = \frac{d}{dz}$$

$$(a+bx)^2 \frac{d^2 y}{dx^2} = b^2 D(D-1)y$$

....

....

$$(a+bx)^n \frac{d^n y}{dx^n} = b^n D(D-1)(D-2) \ldots . (D-n+1)y$$

Using above values in equation (1) we get

$$\frac{d^n y}{dz^n} + a_1 \frac{d^{n-1} y}{dz^{n-1}} + a_2 \frac{d^{n-2} y}{dz^{n-2}} + \cdots + a_n y = Z$$

Which is linear differential with constant coefficients and can be solved by using methods of L. D.

Examples:

1) $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin 2\{\log(1+x)\}$

2) $(2+3x)^2 \frac{d^2y}{dx^2} + 3(2+3x)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

3) $(5+2x)^2 \frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 6x$

4) $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x$

Method of Variation of Parameter:

To solve the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$ by using method of variation of parameter

Where P, Q are constants and X is a function of x only.

Method of solution:

First find complimentary function as $C.F. = c_1u + c_2v$

Where u & v are functions of x only

Now find Wronskain W as $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

& finally find P. I. as

$$P.I. = -u\int \frac{v}{W}X\,dx + v\int \frac{u}{W}X\,dx$$

Then solution is y= C.F. + P. I.

Equations of 1st order & degree higher than first

- **Definition:** An equation of the type

$$\left(\frac{dy}{dx}\right)^n + a_1\left(\frac{dy}{dx}\right)^{n-1} + a_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots \dots \dots + a_{n-1}\left(\frac{dy}{dx}\right) + a_n = 0 \dots \dots .(1)$$

Where $a_1, a_2, \dots \dots . . a_n$ are functions of x & y is called differential equation of 1st order & nth degree.

- If $\frac{dy}{dx} = p$ then above equation can be written as

$$p^n + a_1 p^{n-1} + a_2 p^{n-2} + \dots \dots \dots \dots + a_{n-1} p + a_n = 0$$

- As this equation is of 1st order, its general solution contains only one arbitrary constant.
- To solve the equation (1) we consider the following cases

Equations solvable for P

Equations solvable for x

Equations solvable for y

Clairaut's equation

Case I: Equations solvable for P

The L.H.S. of equation (1) can be factorized into n linear factors in P as

$$[p - f_1(x,y)]\,[p - f_2(x,y)]\,\ldots\ldots\ldots\ldots\,[p - f_n(x,y)] = 0$$

$$i.e.\ [p - f_1(x,y)] = 0,\ [p - f_2(x,y)] = 0\ \ldots\ldots\ldots\ [p - f_n(x,y)] = 0$$

These equations are of 1st order and 1st degree and can be solved easily.

Let, its solution be

$$F_1(x,y,c) = 0, F_2(x,y,c) = 0, \ldots\ \ldots\ldots\ F_n(x,y,c) = 0$$

Hence general solution of equation (1) is given by

$$F_1(x,y,c)\,F_2(x,y,c)\ \ldots\ \ldots\ldots\ F_n(x,y,c) = 0$$

Ex: Solve $p^2 - 5p + 6 = 0$

Solution: Given

$$p^2 - 5p + 6 = 0$$

$$\therefore (p-2)(p-3) = 0$$

$$\therefore p - 2 = 0 \,, p - 3 = 0$$

$$\therefore \frac{dy}{dx} - 2 = 0, \frac{dy}{dx} - 3 = 0$$

solving these equations we get

$$y - 2x - c_1 = 0 \,, y - 3x - c_1 = 0$$

$$\therefore \text{General solution is } (y - 2x - c_1)(y - 3x - c_1) = 0$$

Ex2: Solve

$$x^2p^2 + 3xyp + 2y^2 = 0$$

Solution: Given

$$x^2p^2 + 3xyp + 2y^2 = 0$$

$$\therefore (xp + 2y)(xp + y) = 0$$

$$i.e.\ x\frac{dy}{dx} + 2y = 0 \,, x\frac{dy}{dx} + y = 0$$

Solving we get general solution as

$$(x^2y - c_1)(xy - c_2) = 0$$

Case 2: Equations solvable for y

These equations can be put in the form $y = f(x,p) \dots\dots\dots\dots (1)$

Differentiate (1) w.r.t. x we get $\frac{dy}{dx} = F\left\{x, p, \frac{dp}{dx}\right\}$

$$\therefore p = F\left\{x, p, \frac{dp}{dx}\right\}$$

This is a differential equation involving two variables x and p. suppose its solution be $\emptyset(x,p,c) = 0 \dots\dots\dots\dots (2)$

Now eliminate p from equations (1) and (2) we obtain required solution or solving (1) and (2) for x and y obtain

$$x = \emptyset_1(p,c), y = \emptyset_2(p,c)$$

Where p is a parameter. These two equations together constitute the solution of the given equation.

Ex1: Solve $y = 2px + x^2p^4$

Solution: Given

$$y = 2px + x^2p^4 \ldots\ldots\ldots..(1)$$

Differentiate (1) w. r .t. x and rearranging we get

$$\left(p + 2x\frac{dp}{dx}\right)(1 + 2xp^3) = 0$$

$$\therefore p + 2x\frac{dp}{dx} = 0$$

{ since discarding the factor $(1 + 2xp^3) = 0$

Solving we get $p^2 = c/x$ **.**

Now eliminating p from (1) and (2) we get required solution as

$$y = 2\sqrt{cx} + x^2$$

Case 3: Equations solvable for x

These equations can be put in the form

$$x = f(y, p) \dots \dots \dots \dots (1)$$

Differentiate (1) w. r. t. y we get $\frac{dx}{dy} = F\left\{y, p, \frac{dp}{dy}\right\}$

$$\therefore \frac{1}{p} = F\left\{y, p, \frac{dp}{dy}\right\}$$

This is a differential equation involving two variables y and p. suppose its solution be $\emptyset(y, p, c) = 0 \dots \dots \dots \dots (2)$

Now eliminate p from equations (1) and (2) we obtain required solution or solving (1) and (2) for x and y obtain

$$x = \emptyset_1(p, c), y = \emptyset_2(p, c)$$

Where p is a parameter. These two equations together constitute the solution of the given equation.

Ex1: Solve for x $y = 2px + y^2p^3$

- Solution: Given equation can be written as

$$x = \frac{1}{2}\left\{\frac{y}{p} - y^2p^2\right\} \dots\dots\dots\dots (1)$$

- Differentiate (1) w. r. t. y and rearranging we get

$$\left(p + y\frac{dp}{dy}\right)\left(py + \frac{1}{2p^2}\right) = 0$$

$\therefore p + y\frac{dp}{dy} = 0$ {since discarding the factor $\left(py + \frac{1}{2p^2}\right) = 0$

Solving we get p = c/y ....(2)

Now eliminating p from (1) and (2) we get required solution as

$$y = \frac{2cx}{y} + \frac{c^3}{y}$$

Clairaut's equation:

The equation of the form

$$y = px + f(p)$$

is called Clairaut's equation.

Method of solution:

Since $y = px + f(p) \dots\dots\dots (1)$

Differentiate w. r. t. x

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\therefore p = p + \{x + f'(p)\}\frac{dp}{dx}$$

$$\therefore \{x + f'(p)\}\frac{dp}{dx} = 0$$

$$\therefore \frac{dp}{dx} = 0 \qquad \{ \text{since } [x + f'(p)] \neq 0$$

$$\therefore dp = 0$$

integrating we get

$$p = c \dots\dots\dots\dots (2)$$

Using equation (2) in equation (1) we get

$$y = cx + f(c)$$

This is required solution of equation (1).

Ex3: Solve

$$\sin px \cos y = \cos px \sin y + p$$

Solution: Given equation can be written as

$$\sin px \cos y - \cos px \sin y = p$$

$$\therefore \sin(px - y) = p$$

$$\therefore px - y = \sin^{-1} p$$

$$\therefore y = px - \sin^{-1} p$$

This is in Clairaut's form hence putting p = c we get required solution as

$$\therefore y = cx - \sin^{-1} c$$

Solution Ex 2: Given equation can be written as

$$px - y = e^p$$

$$\therefore y = px - e^p$$

This is in Clairaut's form hence putting p = c we get required solution as

$$\therefore y = cx - e^c$$

Equations reducible to Clairaut's form:

Certain equations can be reduced to clairaut's form by means of suitable substitution.

- **For the equation of the type**

$$y^2 = pxy + f\left(\frac{py}{x}\right)$$

use the substitution $x^2 = u$, $y^2 = v$

- **For the equation of the type**

$$e^{my}(c - mp) = \emptyset\left(pe^{(my-cx)}\right)$$

use the substitution $e^{cx} = u$, $e^{my} = v$

- **For the equation of the type**

$$y = 2px + f(p^2x)$$

use the substitution $x = u^2$, $y = v$

Ex1: Solve $(px-y)(x-py)=2p$ by using the transformation $x^2=u\,,\ y^2=v$

Solution: Given $x^2=u\,,\ y^2=v$

Differentiate w. r. t. x we get

$$2x=\frac{du}{dx}\quad ,2y=\frac{dv}{dx}$$

$$\therefore dx=\frac{du}{2\sqrt{u}}\qquad dy=\frac{dv}{2\sqrt{v}}$$

$$\therefore p=\frac{dy}{dx}=\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$$

Putting this value of p in given equation we get

$$\left\{\frac{u}{\sqrt{v}}\frac{dv}{du}-\sqrt{v}\right\}\left\{\sqrt{u}-\sqrt{u}\,\frac{dv}{du}\right\}=2\,\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$$

Simplifying we get

$$\left\{u\,\frac{dv}{du}-v\right\}\left\{1-\frac{dv}{du}\right\}=2\,\frac{dv}{du}$$

$$\{up_1-v\}\{1-p_1\}=2p_1$$

$$\left\{since\qquad p_1=\frac{dv}{du}\right.$$

$$\therefore v=-\frac{2p_1}{1-p_1}+up_1$$

This is in Clairaut's form, hence putting $p_1 = c$ we get solution as

$$v=uc-\frac{2c}{1-c}$$

$$i.e.\ y^2=x^2c-\frac{2c}{1-c}$$

This is required solution.

Ex2: Solve $e^{3x}(p-1)+p^3e^{2y}=0$

Solution: Here we use

$$e^x = u \quad , e^y = v$$

Differentiate w. r. t. x we get

$$e^x = \frac{du}{dx} \quad , \quad e^y = \frac{dv}{dx}$$

$$\therefore dx = \frac{du}{u} \qquad dy = \frac{dv}{v}$$

$$\therefore p = \frac{dy}{dx} = \frac{u}{v}\frac{dv}{du}$$

Putting this value of p in given equation we get

$$u^3\left(\frac{u}{v}\frac{dv}{du} - 1\right) + \left(\frac{u}{v}\frac{dv}{du}\right)^3 v^2 = 0$$

Simplifying we get

$$v = up_1 + {p_1}^3$$

This is in Clairaut's form, hence putting p_1 = c we get solution as

$$v = uc + c^3$$

$$i.e.\ e^y = e^x c + c^3$$

This is required solution.

Ex3: Solve $(y+px)^2 = py^2$ by substituting $y = u \ ,xy = v$

Solution: We have

$$y = u \quad ,xy = v$$

$$dy = du \quad ,xdy + ydx = dv$$

$$\frac{xdy + ydx}{dy} = \frac{dv}{dy}$$

$$\frac{xdy + ydx}{dy} = \frac{dv}{du}$$

$$x + y\frac{dx}{dy} = p_1$$

$$\therefore p_1 = x + \frac{y}{p}$$

$$\therefore p = \frac{y}{p_1 - x}$$

Putting this value of p in given equation we get

$$\left\{y + \frac{xy}{p_1 - x}\right\}^2 = \frac{y^3}{p_1 - x}$$

Simplifying we get

$$\frac{{p_1}^2}{p_1 - x} = y$$

$$\therefore v = yp_1 - {p_1}^2$$

This is in Clairaut's form, hence putting $p_1 = c$ we get solution as

$$\therefore xy = cy - c^2$$

This is required solution.