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"PARTIAL DIFFERENTIAL EQUATIONS"

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Definition

: An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a partial differential equation

For examples of partial differential equations we list the following

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy \tag{1}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right) \tag{2}$$

$$z\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial y}\right) = x \tag{3}$$

$$\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\right) = xyz \tag{4}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(1 + \left(\frac{\partial z}{\partial y}\right)\right)^{1/2} \tag{5}$$

Order of a Partial Differential Equation:

The order of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation.

above equations (1), (3), (4) are of first order, (5) is of second order and (2) is of the third order.

Degree of a Partial Differential Equation: The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalized, i. e. made free from radicals and fractions so far as derivatives are concerned.

In above equations (1), (2), (3), (4) are of first degree while equation (5) is of second degree.

Linear and Non-linear Partial Differential Equations:

A differential equation is said to be linear if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation is not linear is called a non-linear partial differential equation.

above equations (1) and (4) are linear while equations (2), (3) and (5) are non-linear.

Notations:

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be dependent variable. We adopt the following notations throughout the study of partial differential equations.

$$p = \frac{\partial z}{\partial x}\ , \quad q = \frac{\partial z}{\partial y}\ , \quad r = \frac{\partial^2 z}{\partial x^2}\ , \quad s = \frac{\partial^2 z}{\partial x \partial y}\ , \quad \text{and} \quad t = \frac{\partial^2 z}{\partial y^2} \tag{6}$$

Classification of first order Partial Differential Equations:

Linear equation: A first order equation $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form $P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y)$.

For example $yx^2p + xy^2q = xyz + x^2y^3$ and $p + q = z + xy$. are both first order linear partial differential equations.

Semi-linear equation:

A first order equation $f(x, y, z, p, q) = 0$ is known as a semi-linear equation, if it is linear in p and q and the coefficients of *p*4*and*q*arefunctionsof*x*and*y4 only i. e. if the given equation is of the form $P(x, y)p + Q(x, y)q = R(x, y, z)$.
For example $xyp + x^2yq = x^2y^2z^2$ and $yp + xq = x^2z^2/y^2$ are both first semi-linear partial differential equations.

Quasi-linear equation: A first order equation $f(x, y, z, p, q) = 0$ is known as a quasi-linear equation, if it is linear in p and q, i. e. if the given equation is of the form
$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$.
For example, $x^2zp + y^2zq = xy$ and
$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ are first order quasi-linear partial differential equations.

Non-linear equation: A first order equation $f(x, y, z, p, q) = 0$ which does not come under the above three types, is known as a non-linear equation.

For examples, $p^2 + q^2 = 1$, $pq = z$ and $x^2p^2 + y^2q^2 = z^2$ are all non-linear partial differential equations.

Formation or Origin of Partial Differential Equations: We shall now examine the interesting question of how partial differential equations arise. We show that such equations can be formed by the elimination of arbitrary constants or arbitrary functions.

Derivation of a Partial Differential equation by elimination of arbitrary constants:

Consider an equation

$$F(x, y, z, a, b) = 0 , \tag{7}$$

where a and b denotes arbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (7) with respect to x and y partially, we get

$$\frac{\partial F}{\partial x} + p\left(\frac{\partial F}{\partial z}\right) = 0 , \quad \text{and} \quad \frac{\partial F}{\partial y} + q\left(\frac{\partial F}{\partial z}\right) = 0 \tag{8}$$

Eliminating two arbitrary constants a and b from three equations of (7) and (8), we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0 \;, \tag{9}$$

which is partial differential equation of the first order.

Situation I: When the number of arbitrary constants is less than number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equations of order one.

For example, consider

$$z = ax + b \tag{10}$$

where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (10) partially w. r. t. $'x'$ we get

$$\frac{\partial z}{\partial x} = a \tag{11}$$

Differentiating (10) partially w. r. t. $'y'$ we get

$$\frac{\partial z}{\partial y} = 1 \tag{12}$$

eliminating a between (10) and (12) yields

$$z = x\left(\frac{\partial z}{\partial x}\right) + y \tag{0.1}$$

Since (12) does not contain arbitrary constant, so (12) is also partial differential equation. Thus, we get two partial differential equations.

Situation II: When number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise a unique partial differential equation.

Example: Eliminate a and b from $az + b = a^2x + y$

Solution: Differentiating given equation partially w. r. t. x we get

$$a\left(\frac{\partial z}{\partial x}\right) = a^2 \tag{1}$$

Differentiating given equation partially w. r. t. y we get

$$a\left(\frac{\partial z}{\partial y}\right)=1 \tag{2}$$

eliminating a from (1) and (2), we have

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)=1$$

which is unique partial differential equation of order one.
Situation III: When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential of order usually greater than one.
Example Eliminate a, b and c from $z=ax+by+cxy$
Solution: Given equation is

$$z=ax+by+cxy \tag{1}$$

Differentiating (1) partially w. r. t. x, we have

$$\left(\frac{\partial z}{\partial x}\right) = a + cy \tag{2}$$

Differentiating (1) partially w. r. t. y, we have

$$\left(\frac{\partial z}{\partial y}\right) = b + cx \tag{3}$$

From equations (2) and (3), we have

$$\frac{\partial^2 z}{\partial x^2} = 0 \ , \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = 0 \tag{4}$$

and

$$\frac{\partial^2 z}{\partial x \partial y} = c \tag{5}$$

Now (2) and (3) gives

$$x\left(\frac{\partial z}{\partial x}\right) = ax + cxy \ , \quad \text{and} \quad y\left(\frac{\partial z}{\partial y}\right) = by + cxy \tag{6}$$

$$\Rightarrow x\left(\frac{\partial z}{\partial x}\right)+y\left(\frac{\partial z}{\partial y}\right)=ax+by+cxy+cxy$$

$$\Rightarrow x\left(\frac{\partial z}{\partial x}\right)+y\left(\frac{\partial z}{\partial y}\right)=z+xy\frac{\partial^2 z}{\partial x\partial y}\quad \text{(by using } (1),(5)) \quad (7)$$

Thus, we get two partial differential equations given by (4) and (7), which are all of order two.

Derivation of Partial differential equation by the elimination of arbitrary function ϕ from the equation $\phi(u,v)=0$, where u and v are functions of x, y and z.

Proof: Given

$$\phi(u,v)=0 \quad (1)$$

We treat z as dependent variable and x and y as independent variables so that

$$p=\frac{\partial z}{\partial x}\ ,\quad q=\frac{\partial z}{\partial y}\ ,\quad \frac{\partial y}{\partial x}=0\ ,\quad \text{and}\quad \frac{\partial x}{\partial y}=0 \quad (2)$$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0$$

$$\Rightarrow \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

$$\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}}=-\frac{\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)}{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)} \tag{3}$$

Similarly, differentiating (1) partially w. r. t. y, we get

$$\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}}=-\frac{\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)}{\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)} \tag{4}$$

Eliminating ϕ with the help of (3) and (4), we get

$$\frac{\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)}{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)}=\frac{\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)}{\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)}$$

$$\Rightarrow \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)$$

or

$$Pp + Qq = R \tag{5}$$

where

$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$, $Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$, $R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$.

Thus we obtain a linear partial differential equation of first order and first degree in p and q.

Lagrange's Linear equation:

This equation is of type $Pp + Qq = R$ where P, Q, R are functions of x, y, z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

$$Pp + Qq = R \tag{1}$$

This form of the equation is obtained by eliminating an arbitrary function ϕ from

$$\phi(u, v) = 0 \tag{2}$$

where u and v are functions of x, y, z.

Differentiating (2) partially w. r. t. x and y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0 \qquad (3)$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)=0 \qquad (4)$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (3) and (4), we get
From (3),

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)=-\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) \qquad (5)$$

From (4),

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)=-\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) \qquad (6)$$

Dividing (5) by (6), we get

$$\left[\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right] p + \left[\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right] q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \tag{7}$$

If (1) and (7) are the same, then the coefficients p and q are equal

$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}, \quad Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}, \quad R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \tag{8}$$

Now suppose $u = c_1$ and $v = c_2$ are two solutions, where c_1, c_2 are constants.

Differentiating $u = c_1$ and $v = c_2$, we get

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \tag{9}$$

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \tag{10}$$

Solving (9) and (10), we get

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \tag{11}$$

From (8) and (11), we have $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
Solutions of these equations are $u = c_1$ and $v = c_2$
Hence, $f(u, v) = 0$ is the required solution of (1).

Partial differential Equations Non-linear in p and q:

Type I: $f(p, q) = 0$
Type II: $f(z, p, q) = 0$
Type III: $f_1(x, p) = f_2(y, q)$
Type IV: $z = px + qy + f(p, q)$

Type I: $f(p,q)=0$

Method:

Let the required solution be

$$z = ax + by + c \tag{1}$$

Differentiating partially w. r, t. x and y, we get

$$\frac{\partial z}{\partial x} = p = a, \quad \frac{\partial z}{\partial y} = q = b$$

On putting the values of p and q in the given equation, we have

$$f(a,b) = 0$$

From this, find the value of b in terms of a and substitute in equation (1), that will be the required solution.
Example: Solve $p^2 + q^2 = 1$

Solution:

Let the complete integral be

$$z = ax + by + c \tag{1}$$

Differentiating partially w. r, t. x and y, we get

$$\frac{\partial z}{\partial x} = p = a, \quad \frac{\partial z}{\partial y} = q = b$$

On putting the values of p and q in the given equation, we have

$$a^2 + b^2 = 1$$
$$\Rightarrow b = \pm\sqrt{1 - a^2}$$

Putting this value of b in (1), we get

$$z = ax \pm \sqrt{1 - a^2}y + c \tag{0.2}$$

which required complete integral.

Type II: $f(p,q,z)=0$

Method:

Let z be a function of X where

$$X = x + ay \tag{1}$$

Differentiating (1) partially w. r, t. x and y, we get

$$\frac{\partial z}{\partial x} = p = \frac{\partial z}{\partial X}, \quad \frac{\partial z}{\partial y} = q = a\frac{\partial z}{\partial X}$$

On putting the values of p and q in the given equation, it becomes $f\left(z, \frac{\partial z}{\partial X}, a\frac{\partial z}{\partial X}\right) = 0$ which is an ordinary differential equation of first order. Solve it and resubstitute $X = x + ay$ gives required solution.

Example: Solve $p(1+q) = qz$

Solution: We have

$$p(1+q) = qz \tag{1}$$

Put $z = f(X)$, where $X = x + ay$ and $p = \frac{\partial z}{\partial X}$, $q = a\frac{\partial z}{\partial X}$

Hence, equation (1) becomes

$$\frac{\partial z}{\partial X}\left(1 + a\frac{\partial z}{\partial X}\right) = az\frac{\partial z}{\partial X} \Rightarrow 1 + a\frac{\partial z}{\partial X} = az$$
$$a\frac{\partial z}{\partial X} = az - 1 \Rightarrow \frac{\partial z}{\partial X} = \frac{az-1}{a}$$
$$\frac{\partial X}{\partial z} = \frac{a}{az-1} \Rightarrow dX = \frac{adz}{az-1}$$
$$\Rightarrow X = log(az-1) + logc$$
$$\Rightarrow x + ay = logc(az-1)$$

which is required solution.

Type III: Equation of the type $f_1(x,p) = f_2(y,q)$

Method: Equating each side to constant a

$$f_1(x,p) = f_2(y,q) = a \Rightarrow f_1(x,p) = a, \quad f_2(y,q) = a$$
$$\Rightarrow p = f_1(x,a), \quad q = f_2(y,a)$$

Since $\quad dz = pdx + qdy \Rightarrow dz = f_1(x, a)dx + f_2(y, a)dy$

integrating, we get

$$z = \int f_1(x, a)dx + \int f_2(y, a)dy + c$$

Example: Solve $p - x^2 = q + y^2$
Solution: Equating each side to constant a, we have

$$p - x^2 = a, \quad q + y^2 = a \Rightarrow p = x^2 + a, \quad q = a - y^2$$

putting these values of p and q in

$$dz = pdx + qdy \Rightarrow dz = (x^2 + a)dx + (a - y^2)dy$$

integrating, we get

$$z = \frac{x^3}{3} + ax + ay - \frac{y^3}{3} + c$$

which is required solution.

Type IV: Equation of the type $z = px + qy + f(p, q)$

Its solution is $z = ax + by + f(a, b)$

Example: Solve $z = px + qy + p^2 + q^2$

Solution: Given

$$z = px + qy + p^2 + q^2$$

which is in Clairaut's form hence, putting $p = a$ and $q = b$ we get required solution as

$$z = ax + by + a^2 + b^2$$

Example: Solve $z = px + qy + 2\sqrt{pq}$

Solution: Given

$$z = px + qy + 2\sqrt{pq}$$

which is in Clairaut's form hence, putting $p = a$ and $q = b$ we get required solution as

$$z = ax + by + 2\sqrt{ab}$$

Type V: Charpit's Method:

General method of solving partial differential differential equation with two independent variables.

Solution: Let the general partial differential equation be

$$f(x, y, z, p, q) = 0 \tag{1}$$

Since z depends on x, y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$dz = pdx + qdy \tag{2}$$

The main thing in Charpit's method is to find another relation between the variables x, y, z and p, q. Let the relation be

$$\phi(x, y, z, p, q) = 0 \tag{3}$$

On solving (1) and (3), we get the values of p and q.
These values of p and q when substituted in (2) it becomes integrable.
To determine ϕ, (1) and (3) are differentiated w. r. t. x and y giving

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}p + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad (4)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}p + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad (5)$$

Eliminating $\frac{\partial p}{\partial x}$ between the equation (4), we have

$$\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial p}-\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial p}\right)+p\left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial p}-\frac{\partial \phi}{\partial z}\frac{\partial f}{\partial p}\right)+ \\ +\frac{\partial q}{\partial x}\left(\frac{\partial f}{\partial q}\frac{\partial \phi}{\partial p}-\frac{\partial \phi}{\partial q}\frac{\partial f}{\partial p}\right)=0 \qquad (6)$$

On eliminating $\frac{\partial q}{\partial y}$ between the equation (5), we have

$$\left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial q}-\frac{\partial \phi}{\partial y}\frac{\partial f}{\partial q}\right)+q\left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial q}-\frac{\partial \phi}{\partial z}\frac{\partial f}{\partial q}\right)+ \\ +\frac{\partial p}{\partial y}\left(\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial q}-\frac{\partial \phi}{\partial p}\frac{\partial f}{\partial q}\right)=0 \qquad (7)$$

Adding (6) and (7) and keeping in view the relation $\frac{\partial p}{\partial y}=\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial q}{\partial x}$, the terms of the last brackets of (6) and (7) cancel. On rearranging, we get

$$\frac{\partial \phi}{\partial p}\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)+\frac{\partial \phi}{\partial q}\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)+\frac{\partial \phi}{\partial z}\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)+ + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y}=0 \qquad (8)$$

Equation (8) is a Lagrange's linear equation of the first order with x, y, z, p, q as independent variables and ϕ as dependent variable. Its subsidiary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{d\phi}{0} \qquad (9)$$

Any of the integral of (9) satisfies (8). Such an integral involving p or q or both may be taken as assumes relation (3). However, we should choose the simplest integral involving p and q derived from (9). This relation and equation (1) gives values of p and q. The values of p and q are substituted in (2). On integrating gives required solution.

THANK YOU