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BALASAHEB DESAI COLLEGE, PATAN

"BETA AND GAMMA FUNCTIONS, D. U. I. S. , MULTIPLE INTEGRALS" AND FOURIER SERIES

By

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Gamma Function:

Definition:

Consider the definite integral $\int_0^\infty e^{-x}x^{n-1}dx$, it is denoted by the symbol Γn (we read it as Gamma $'n'$) and is called as Gamma function of n. Thus,

$$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx \quad (n > 0) \tag{1}$$

Gamma function is also called as Euler's integral of the second kind.

Properties of Gamma Function:

1. $\Gamma 1 = 1, \quad \Gamma 0 = \infty$
2. $\Gamma(n+1) = n\Gamma n$
3. $\Gamma(n+1) = n!$
4. $\Gamma(1/2) = \sqrt{\pi}$
5. $\Gamma n = 2\int_0^\infty e^{-x^2}x^{2n-1}dx$
6. $\Gamma n = k^n\int_0^\infty e^{-ky}y^{n-1}dy$

Proofs:

5.

We have,

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

put $x = t^2$, $\therefore dx = 2tdt$, when $x = 0$ we get $t = 0$ and when $x = \infty$ we get $t = \infty$

$$\therefore \Gamma n = \int_0^{\infty} e^{-t^2} t^{2n-2} 2tdt = 2\int_0^{\infty} e^{-t^2} t^{2n-1} dt$$

$$\therefore \Gamma n = 2\int_0^{\infty} e^{-x^2} x^{2n-1} dx$$

6. We have,

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

put $x = ky$, $\therefore dx = kdy$, when $x = 0$ we get $t = 0$ and when $x = \infty$ we get $y = \infty$

$$\therefore \Gamma n = \int_0^\infty e^{-ky} k^{n-1} y^{n-1} k dy = k^n \int_0^\infty e^{-ky} y^{n-1} dy$$

$$\therefore \Gamma n = k^n \int_0^\infty e^{-ky} y^{n-1} dy$$

Examples: Evaluate

1. $\int_0^\infty x^{1/4} e^{-\sqrt{x}} dx$
2. $\int_0^\infty 3^{-4x^2} dx$
3. $\int_0^1 x^{\alpha-1} \left(log \frac{1}{x}\right)^{n-1} dx, \quad (\alpha > 0)$
4. $\int_0^1 (xlogx)^4 dx$
5. $\int_0^\infty \frac{x^a}{a^x} dx$

Beta Function:

Definition:

Consider the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx, m > 0, n > 0$. It is denoted by the symbol $B(m, n)$.

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, m > 0, n > 0$$

The Beta function is also called as Euler's integral of the first kind.

Properties of Beta Function:

1. $B(m, n) = B(n, m)$
2. $B(m.n) = 2\int_0^{\pi/2} sin^{2m-1}\theta cos^{2n-1}\theta d\theta$

Proof: By definition

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Put $x = sin^2\theta$, $\therefore dx = 2sin\theta cos\theta d\theta$, when $x = 0$ we get $\theta = 0$ and when $x = 1$ we get $\theta = \pi/2$

$$B(m,n) = \int_0^{\pi/2} sin^{2m-2}\theta(1 - sin^2\theta)^{2n-2} 2sin\theta cos\theta d\theta$$

$$\therefore B(m.n) = 2\int_0^{\pi/2} sin^{2m-1}\theta cos^{2n-1}\theta d\theta$$

We consider this as a definition of Beta function.
Further let $2m - 1 = p$, $2n - 1 = q \therefore m = \frac{p+1}{2}, n = \frac{q+1}{2}$

$$B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} sin^p\theta cos^q\theta d\theta$$

$$\therefore \int_0^{\pi/2} sin^p\theta cos^q\theta d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Note: Students are advised to remember this formula.

3. $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

By definition

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx.$$

put $x = \frac{t}{1+t}$, $\therefore t = \frac{x}{1-x} \Rightarrow dx = \frac{1}{(1+t)^2}dt.$
When $x = 0$, $t = 0$ and when $x = 1$, $t = \infty$

$$B(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m-1}}(1 - \frac{t}{1+t})^{n-1} \cdot \frac{dt}{(1+t)^2}$$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m-1}(1+t)^{n-1}(1+t)^2} = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt$$

$$\therefore B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx.$$

4.Relation between Beta and Gamma function
$B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$**:**

Proof:

We have

$$\Gamma m = 2\int_0^\infty e^{-x^2} x^{2m-1} dx \tag{1}$$

Similarly

$$\begin{aligned}\Gamma n =& 2\int_0^\infty e^{-y^2} y^{2n-1} dy \\ \therefore \Gamma m \Gamma n =& 4\int_0^\infty e^{-x^2} x^{2m-1} dx \int_0^\infty e^{-y^2} y^{2n-1} dy \\ =& 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dxdy \end{aligned} \tag{2}$$

Put $x = rcos\theta, y = rsin\theta \therefore dxdy = rdrd\theta$. To cover the region in (2) which is the entire first quadrant, r varies from 0 to ∞ and θ varies from 0 to $\pi/2$.

$$\therefore \Gamma m \Gamma n = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2(m+n)-1} cos^{2m-1}\theta sin^{2n-1}\theta drd\theta$$

$$\Gamma m \Gamma n = \left[2 \int_0^{\pi/2} cos^{2m-1} \theta sin^{2n-1} \theta d\theta \right] \left[2 \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr \right] \tag{3}$$

But by (2),

$$\left[2 \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr \right] = \Gamma(m+n)$$

and by property of Beta function

$$\left[2 \int_0^{\pi/2} cos^{2m-1} \theta sin^{2n-1} \theta d\theta \right] = B(m,n)$$

Thus (3) gives $\Gamma m \Gamma n = B(m,n)\Gamma(m+n)$.

$$B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

Duplication Formula of Gamma Function:

$$2^{2m-1}\Gamma m\Gamma(m+1/2)=\sqrt{\pi}\Gamma(2m)$$

Proof: Consider

$$\frac{1}{2}\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q+2}{2})}=\int_0^{\pi/2} sin^p\theta cos^q\theta d\theta$$

Put $p=2m-1,\ q=2m-1$. i. e. $\frac{p+1}{2}=m, \frac{q+1}{2}=m$

$$\frac{1}{2}\frac{\Gamma m\Gamma n}{\Gamma(2m)}=\int_0^{\pi/2} sin^{2m-1}\theta cos^{2n-1}\theta d\theta$$

$$\frac{\Gamma m\Gamma m}{\Gamma(2m)}=\frac{2}{2^{2m-1}}\int_0^{\pi/2}(2sin\theta cos\theta)^{2m-1}d\theta$$

$$=\frac{2}{2^{2m-1}}\int_0^{\pi/2}(sin2\theta)^{2m-1}d\theta$$

Put $2\theta=t$, $\therefore d\theta=\frac{1}{2}dt$ and when $\theta=0, t=0$ and $\theta=\pi/2, t=\pi$

$$\begin{aligned}\frac{\Gamma m \Gamma m}{\Gamma(2m)} =& \frac{1}{2^{2m-1}}\int_0^{\pi}(sint)^{2m-1}dt = \frac{2}{2^{2m-1}}\int_0^{\pi/2} sin^{2m-1}tdt \\ =& \frac{2}{2^{2m-1}}\int_0^{\pi/2} sin^{2m-1}tcos^0 tdt \\ =& \frac{2}{2^{2m-1}}\frac{1}{2}\frac{\Gamma\frac{(2m-1+1)}{2}\Gamma\frac{0+1}{2}}{\Gamma\frac{2m-1+0+2}{2}} \\ =& \frac{1}{2^{2m-1}}\frac{\Gamma m\sqrt{\pi}}{\Gamma(m+1/2)} \\ & \therefore 2^{2m-1}\Gamma m\Gamma(m+1/2) = \sqrt{\pi}\Gamma(2m)\end{aligned}$$

Note: Putting $m = 1/4$ in above formula, we get

$$\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$$

Show that

$$\int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}dy = B(m,n)(\text{for proof consider 3rd def})$$

D. U. I. S

Introduction:

There are some definite integrals which cannot be obtained by methods studied so far. However, we can differentiate the given function under the integral sign and form the resulting function we can obtain the required integral. This is known as differentiation under integral sign abbreviated as D. U. I. S.

Rule 1: If $f(x, \alpha)$ is a continuous function of x and α is a parameter and if $\frac{\partial f}{\partial \alpha}$ is a continuous function of x and α together throughout the integral [a,b] where a, b are constants and independent of α and if

$$I(\alpha) = \int_a^b f(x, \alpha) dx$$

then

$$I'(\alpha) = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha) dx$$

Proof:

We have

$$I(\alpha) = \int_a^b f(x,\alpha)dx$$

$$\therefore I(\alpha+\delta\alpha) = \int_a^b f(x,\alpha+\delta\alpha)dx$$

$$\therefore I(\alpha+\delta\alpha) - I(\alpha) = \int_a^b \left[f(x,\alpha+\delta\alpha) - f(x,\alpha)\right]dx$$

$$\therefore \frac{I(\alpha+\delta\alpha) - I(\alpha)}{\delta\alpha} = \int_a^b \frac{\left[f(x,\alpha+\delta\alpha) - f(x,\alpha)\right]}{\delta\alpha}dx$$

Taking limit of both sides as $\delta\alpha \to 0$, we have,

$$I'(\alpha) = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)dx$$

Rule 2: (Leibnitzs Rule)

If $f(x, \alpha)$ is a continuous function of x and α is a parameter and if $\frac{\partial f}{\partial \alpha}$ is a continuous function of x and α together throughout the integral [a,b] where $a(\alpha), b(\alpha)$ are functions of α and if

$$I(\alpha) = \int_a^b f(x, \alpha) dx$$

then

$$I'(\alpha) = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha) dx + f(b, \alpha) \frac{db}{d\alpha} - f(a, \alpha) \frac{da}{d\alpha}$$

Examples: Evaluate

1. $\displaystyle\int_0^1 \frac{x^{\alpha - 1}}{logx} dx$
2. $\displaystyle\int_0^\infty \frac{e^{-x}}{x} (1 - e^{-ax}) dx, \quad (a > -1)$

Multiple Integrals:

Evaluation of Double Integrals:

Double integrals over a region 'R' may be evaluated by two successive integrations as follows

1. Suppose R can be described by $x = a, x = b, y = f_1(x)$ and $y = f_2(x)$, then

$$I = \int\int f(x,y)dydx = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)dy\right] dx \qquad (1)$$

We first integrate the inner integral w. r. t. y keeping x as constant between the limits $y = f_1(x), y = f_2(x)$ and the resulting expression, say $g(x)$, be integrated w. r. t. x between the limits $x = a, x = b$. We then obtain the value of double integral in (1).

2. Suppose that R can be described by $y = c, y = d, x = f_1(y)$ and $x = f_2(y)$, then

$$I = \int\int f(x,y)dxdy = \int_c^d \left[\int_{f_1(y)}^{f_2(y)} f(x,y)dx\right] dy \qquad (2)$$

We first integrate the inner integral w. r. t. x keeping y as constant between the limits $x = f_1(y), x = f_2(y)$ and the resulting expression, say $h(y)$, be integrated w. r. t. y between the limits $y = c, y = d$. We then obtain the value of double integral in (2).

3. Suppose R can be described by $x = a, x = b, y = c$, and $y = d$, then

$$I = \int_a^b \left[\int_c^d f(x,y)dy \right] dx \ \text{ OR} I = \int_c^d \left[\int_a^b f(x,y)dx \right] dy$$

Note: When a, b, c, d are constants then, the order of integration must be clearly specified.

Problems on integrals when the limits are not provided: If the limits of integral are not provided but the region of integration is given, then we first find the points of integration of the curves and draw the given region.

Note: Vertical strip (or horizontal strip) is to be chosen, in order to integrate w. r. t y (or x) first, in such a way that the evaluation of inner integral becomes possible and simple.

Problems on change of order of integration:

It may happen that the integrand $f(x, y)$ in the double integral $\int[\int f(x, y)dy]dx$ is difficult, or even impossible to integrate w. r. t. y first, but can be easily integrated w. r.t. x first, In such an event, it becomes necessary to reverse (i. e. change) the order of integration in the double integral i. e. to work it out in the form $\int[\int f(x, y)dx]dy$.
Similarly, in $I = \int[\int f(x, y)dx]dy$, if it is difficult to integrate first w. r. t. x, then, by changing the order of integration, we get, $I = \int[\int f(x, y)dy]dx$.

How to change the order of integration?

Method I: Let the given integral be

$$I = \int_{x=a}^{x=b} \left[\int_{y=f_1(x)}^{y=f_2(x)} f(x, y)dy \right] dx$$

Step I: Given region R is bounded by $x = a, x = b$, $y = f_1(x), y = f_2(x)$ and find points of intersections.

Sketch the region of integration R and check whether it is correct or wrong by drawing a vertical strip.

Step II: Now to reverse the order, draw a horizontal strip cutting through region R. Write down the given integral I with order of integration reversed as

$$I = \int \left[\int f(x, y) dx \right] dy$$

Step III: Find the new limits Limits for x: A function of y, say $x = g_1(y) tox = g_2(y)$ Limits for y: constants, say $y = ctoy = d$ Hence by changing the order of integration,

$$I = \int_{c}^{d} \left[\int_{g_1(y)}^{g_2(y)} f(x, y) dx \right] dy$$

Method II: Let the given integral be

$$I = \int_{y=c}^{y=d} \left[\int_{x=g_1(y)}^{x=g_2(y)} f(x, y) dx \right] dy$$

Step I: Given region R is bounded by $y = c, y = d$, $x = g_1(y), x = g_2(y)$ and find points of intersections.

Sketch the region of integration R and check whether it is correct or wrong by drawing a horizontal strip.
Step II: Now to reverse the order, draw a vertical strip cutting through region R. Write down the given integral I with order of integration reversed as

$$I = \int \left[\int f(x, y) dy \right] dx$$

Step III: Find the new limits Limits for y: A function of x, say $y = f_1(x) toy = f_2(x)$ Limits for y: constants, say $x = atox = b$
Hence by changing the order of integration,

$$I = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x, y) dy \right] dx$$

Transformation of Cartesian Double Integral into Polar Double Integral:

In many cases, it is quite advantageous to convert $\int\int f(x,y)dxdy$ into polar form by transformations $x = rcos\theta, y = rsin\theta$. Function $f(x,y)$ gets converted to$F(r,\theta)$, equations of bounding Cartesian curves are expressed in polar form. The area element $dxdy$ is replaced by area element $rdrd\theta$.
To find limits in polar double integral consider radical strip in the region, rotate the radical strip in anti clock wise direction gives limits for θ and on the extreme points of the strip find limits for r. Now the integral becomes

$$I = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=f_1(\theta)}^{r=f_2(\theta)} F(r.\theta)drd\theta$$

Now first integrate inside integral w. r. t. r treating θ as constant within the limits $r = f_1(\theta)$ and $r = f_2(\theta)$ further the resulting integral integrate w. r. t.θ within the limits $\theta = \alpha$ and $\theta = \beta$, gives the required value of double integral.

Note:

If the given region of double integral is bounded by circle or integrad involves expression of the type $(x^2 + y^2)^{n/2}$ change the given double integral to polar coordinates and evaluate.

Evaluation of Triple Integration:

Use of Notation:

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)dzdydx =$$

$$= \int_a^b \left\{ \int_{f_1(x)}^{f_2(x)} \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)dz \right] dy \right\} dx$$

Here solve inner integral first w. r. t. z then w. r. t. y and outermost integral finally w. r. t. x.

Note:

The order of integration depends upon the distribution of limits.

FOURIER SERIES

Introduction: In many physical and engineering problems, particularly those connected with vibration and conduction of heat, it is more useful to be able to express a real valued function in series of sines and cosines. Most of the single valued periodic functions which occur in applied mathematics can be expressed in the form

$$\frac{a_0}{2} + a_1 cosx + a_2 cos2x + ... + a_n cosnx + ... +$$
$$+ b_1 sinx + b_2 sin2x + ... + b_n sinnx + ...$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx$$

is called the Fourier series, where $a_0, a_1, a_2, ..., a_n, ..., b_1, b_2, ..., b_n, ...$ are constants.

USEFUL INTEGRALS

The following integrals are useful in Fourier Series.

$(i) \int_0^{2\pi} sinnxdx = 0,$ $(ii) \int_0^{2\pi} cosnxdx = 0,$

$(iii) \int_0^{2\pi} sin^2 nxdx = \pi,$ $(iv) \int_0^{2\pi} cos^2 nxdx = \pi,$

$(v) \int_0^{2\pi} sinnx \cdot sinmxdx = 0,$ $(vi) \int_0^{2\pi} cosnx \cdot cosmxdx = 0,$

$(vii) \int_0^{2\pi} sinnx \cdot cosmxdx = 0,$ $(viii) \int_0^{2\pi} sinnx \cdot cosnxdx = 0,$

$(ix) sinn\pi = 0, \quad cosn\pi = (-1)^n$ where $n \in I$

$(x) \int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$

Determination of Fourier Coefficients (Euler's Formulae)

$$f(x) = \frac{a_0}{2} + a_1 cosx + a_2 cos2x + ... + a_n cosnx + ... + \\ + b_1 sinx + b_2 sin2x + ... + b_n sinnx + ... \qquad (1)$$

(i) **To find** a_0 : Integrate both sides of (1) from $x = 0$ to $x = \alpha + 2\pi$

$$\int_{\alpha}^{\alpha+2\pi} f(x)dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} dx + a_1 \int_{\alpha}^{\alpha+2\pi} cosxdx + \\ + a_2 \int_{\alpha}^{\alpha+2\pi} cos2xdx + ... + a_n \int_{\alpha}^{\alpha+2\pi} cosnxdx + ... + \\ + b_1 \int_{\alpha}^{\alpha+2\pi} sinxdx + b_2 \int_{\alpha}^{\alpha+2\pi} sin2xdx + ... + \\ + b_n \int_{\alpha}^{\alpha+2\pi} sinnxdx + ...$$

$$\int_{\alpha}^{\alpha+2\pi} f(x)dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} dx \quad (\text{other int.} = 0 \;\text{ by }\; (i) \;\text{ and }\; (ii))$$

$$= \frac{a_0}{2}2\pi \qquad \Rightarrow a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx$$

To find a_n: multiply each side of (1) by *cosnx* and integrate from $x = 0$ to $x = \alpha + 2\pi$

$$\int_{\alpha}^{\alpha+2\pi} f(x)cosnxdx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} cosnxdx + a_1\int_{\alpha}^{\alpha+2\pi} cosnxcosxdx +$$
$$+ a_2\int_{\alpha}^{\alpha+2\pi} cosnxcos2xdx + ... + a_n\int_{\alpha}^{\alpha+2\pi} cos^2nxdx + ... +$$
$$+ b_1\int_{\alpha}^{\alpha+2\pi} cosnxsinxdx + b_2\int_{\alpha}^{\alpha+2\pi} cosnxsin2xdx + ... +$$
$$+ b_n\int_{\alpha}^{\alpha+2\pi} cosnxsinnxdx + ...$$

$$\int_{\alpha}^{\alpha+2\pi} f(x)cosnxdx = a_n \int_{\alpha}^{\alpha+2\pi} cos^2 nxdx$$
$$= a_n\pi \quad \text{(other int.=0 by above formulae)}$$
$$\therefore \quad a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)cosnxdx$$

By taking $n = 1, 2, 3, ...$ we can find the values of $a_1, a_2, ...$

To find b_n: multiply each side of (1) by *sinnx* and integrate from $x = 0$ to $x = \alpha + 2\pi$

$$\int_{\alpha}^{\alpha+2\pi} f(x)sinnxdx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} sinnxdx + a_1\int_{\alpha}^{\alpha+2\pi} sinnxcosxdx +$$
$$+ a_2\int_{\alpha}^{\alpha+2\pi} sinnxcos2xdx + ... + a_n\int_{\alpha}^{\alpha+2\pi} sinnxcosnxdx + ... +$$
$$+ b_1\int_{\alpha}^{\alpha+2\pi} sinnxsinxdx + b_2\int_{\alpha}^{\alpha+2\pi} sinnxsin2xdx + ... +$$
$$+ b_n\int_{\alpha}^{\alpha+2\pi} sin^2 nxdx + ...$$

$$\int_{\alpha}^{\alpha+2\pi} f(x) sinnx dx = b_n \int_{\alpha}^{\alpha+2\pi} sin^2 nx dx$$

$$= b_n \pi \quad \text{(other int.=0 by above formulae)}$$

$$\therefore \quad b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) sinnx dx$$

By taking $n = 1, 2, 3, ...$ we can find the values of $b_1, b_2, ...$

Corol 1. When $\alpha = 0$, the interval becomes $0 \leq x \leq 2\pi$ and above formulae becomes

$$a_0 = \frac{1}{\pi} \int_{0}^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) cosnx dx$$

$$b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) sinnx dx$$

Corol 2.:

When $\alpha = -\pi$, the interval becomes $-\pi \leq x \leq \pi$ and above formulae becomes

$$
\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)cosnxdx \qquad \text{(A)} \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)sinnxdx
\end{aligned}
$$

Fourier Series in Change of Interval:

$$
f(x) = \frac{a_0}{2} + a_1 cos(\frac{\pi x}{c}) + a_2 cos(\frac{2\pi x}{c}) + ... + a_n cos(\frac{n\pi x}{c}) + ... + \\
+ b_1 sin(\frac{\pi x}{c}) + b_2 sin(\frac{2\pi x}{c}) + ... + b_n sin(\frac{n\pi x}{c}) + ... \qquad (1)
$$

(i) **To find** a_0 : Integrate both sides of (1) from $x = 0$ to $x = \alpha + 2c$

$$\int_{\alpha}^{\alpha+2c} f(x)dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2c} dx + a_1\int_{\alpha}^{\alpha+2c} cos(\frac{\pi x}{c})dx +$$
$$+ a_2\int_{\alpha}^{\alpha+2c} cos(\frac{2\pi x}{c})dx + ... + a_n\int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})dx +$$
$$+ ... + b_1\int_{\alpha}^{\alpha+2c} sin(\frac{\pi x}{c})dx + b_2\int_{\alpha}^{\alpha+2c} sin(\frac{2\pi x}{c})dx +$$
$$+ ... + b_n\int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})dx + ...$$

$$\int_{\alpha}^{\alpha+2c} f(x)dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2c} dx \quad \text{(other int. = 0 by (i) and (ii))}$$
$$= \frac{a_0}{2}2c \quad \Rightarrow a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)dx$$

To find a_n:

multiply each side of (1) by $cos(\frac{n\pi x}{c})$ and integrate from $x = 0$ to $x = \alpha + 2c$

$$\int_{\alpha}^{\alpha+2c} f(x)cos(\frac{n\pi x}{c})dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2c} cos(\frac{\pi x}{c})dx +$$

$$+ a_1 \int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})cos(\frac{\pi x}{c})dx +$$

$$+ a_2 \int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})cos(\frac{2\pi x}{c})dx + ... +$$

$$+ a_n \int_{\alpha}^{\alpha+2c} cos^2(\frac{n\pi x}{c})dx + ... +$$

$$+ b_1 \int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})sin(\frac{\pi x}{c})dx +$$

$$+ b_2 \int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})sin(\frac{2\pi x}{c})dx + ... +$$

$$+ b_n \int_{\alpha}^{\alpha+2c} cos(\frac{n\pi x}{c})sin(\frac{n\pi x}{c})dx + ...$$

$$\int_{\alpha}^{\alpha+2c} f(x)cos(\frac{n\pi x}{c})dx = a_n \int_{\alpha}^{\alpha+2c} cos^2(\frac{n\pi x}{c})dx$$
$$= a_n c \quad \text{(other int.=0 by above formulae)}$$
$$\therefore \quad a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)cos(\frac{n\pi x}{c})dx$$

To find b_n: multiply each side of (1) by $sin(\frac{n\pi x}{c})$ and integrate from $x = 0$ to $x = \alpha + 2c$

$$\int_{\alpha}^{\alpha+2c} f(x)sin(\frac{n\pi x}{c})dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})dx+$$
$$+ a_1 \int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})cos(\frac{\pi x}{c})dx+$$

$$+ a_2 \int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})cos(\frac{2\pi x}{c})dx + ...+$$

$$+ a_n \int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})cos(\frac{n\pi x}{c})dx + ...+$$

$$+ b_1 \int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})sin(\frac{\pi x}{c})dx+$$

$$+ b_2 \int_{\alpha}^{\alpha+2c} sin(\frac{n\pi x}{c})sin(\frac{2\pi x}{c})dx + ...+$$

$$+ b_n \int_{\alpha}^{\alpha+2c} sin^2(\frac{n\pi x}{c})dx + ...$$

$$\int_{\alpha}^{\alpha+2c} f(x)sin(\frac{n\pi x}{c})dx = b_n \int_{\alpha}^{\alpha+2c} sin^2(\frac{n\pi x}{c})dx$$

$$= b_n c \quad \text{(other int.=0 by above formulae)}$$

$$\therefore \quad b_n = \frac{1}{c} \int_{\alpha}^{\alpha+2c} f(x)sin(\frac{n\pi x}{c})dx$$

Corol. 1:

When $\alpha = 0$, the interval becomes $0 \leq x \leq 2c$ and above formulae becomes

$$a_0 = \frac{1}{c}\int_0^{2c} f(x)dx$$

$$a_n = \frac{1}{c}\int_0^{2c} f(x)cos(\frac{n\pi x}{c})dx$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)sin(\frac{n\pi x}{c})dx$$

Corol. 2: When $\alpha = -c$, the interval becomes $-c \leq x \leq c$ and above formulae becomes

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)cos(\frac{n\pi x}{c})dx$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)sin(\frac{n\pi x}{c})dx$$

Even and Odd Functions:

When $f(x)$ posses certain symmetrical properties about origin, the determination of coefficients in its Fourier expansion becomes quite simple.

For the function $f(x)$ to be odd or even, it must be defined in an interval, where origin is the mid-point of the interval i. e. $f(x)$ must have interval $-\pi < x < \pi$.

Even Function: $f(x)$ is said to be an even function in $-\pi < x < \pi$, if $f(-x) = f(x)$.

Thus functions $x^2, cosx, xsinx$ are even functions in $-\pi < x < \pi$.

Graphically an even function is symmetric about y-axis. Thus for even function $f(x)$ defined in the interval $-\pi \leq x \leq \pi$, the Fourier coefficients a_0, a_n, b_n given in (A) are reduced as follows.

$$
\begin{aligned}
a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
=& \frac{1}{\pi}\left[\int_{0}^{\pi} f(x)dx + \int_{0}^{\pi} f(-x)dx\right] \quad \text{(Since by integral theorem)}
\end{aligned}
$$

$$
\begin{aligned}
a_0 =& \frac{1}{\pi}\left[\int_0^{\pi} f(x)dx + \int_0^{\pi} f(x)dx\right] \quad (\text{Since} \quad f(-x) = f(x) \\
=& \frac{2}{\pi}\int_0^{\pi} f(x)dx
\end{aligned}
$$

and

$$
\begin{aligned}
a_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x)cosnxdx + \int_0^{\pi} f(-x)cosn(-x)dx\right] \; (\text{Since by int. theore} \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x)cosnxdx + \int_0^{\pi} f(x)cosnxdx\right] \quad (\text{Since} \quad f(-x) = f(x) \\
=& \frac{2}{\pi}\int_0^{\pi} f(x)cosnxdx
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
b_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sinnx dx \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x) sinnx dx + \int_0^{\pi} f(-x) sinn(-x) dx\right] \quad \text{(Since by int. theorem} \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x) sinnx dx - \int_0^{\pi} f(x) sinnx dx\right] \quad \text{(Since } \; f(-x) = f(x) \\
=& 0
\end{aligned}
$$

Odd Function: $f(x)$ is said to be an odd function in $-\pi < x < \pi$, if $f(-x) = -f(x)$. Thus functions $x^3, sinx, x^2 sinx$ etc are odd functions in $-\pi < x < \pi$. Graphically an odd function is symmetric about origin. Thus for odd function $f(x)$ defined in the interval $-\pi \leq x \leq \pi$, the Fourier coefficients a_0, a_n, b_n given in (A) are reduced as follows.

$$
\begin{aligned}
a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x) dx + \int_0^{\pi} f(-x) dx\right] \quad \text{(Since by integral theorem)}
\end{aligned}
$$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} f(x)dx - \int_0^{\pi} f(x)dx\right] \quad (\text{Since} \quad f(-x) = -f(x)$$
$$= 0$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$
$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)cosnxdx + \int_0^{\pi} f(-x)cosn(-x)dx\right] \; (\text{Since by int. theore}$$
$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)cosnxdx - \int_0^{\pi} f(x)cosnxdx\right] \quad (\text{Since} \quad f(-x) = -f(x)$$
$$= 0$$

Similarly,

$$
\begin{aligned}
b_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sinnxdx \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x) sinnxdx + \int_0^{\pi} f(-x) sinn(-x)dx\right] \text{ (Since by int. theorer} \\
=& \frac{1}{\pi}\left[\int_0^{\pi} f(x) sinnxdx + \int_0^{\pi} f(x) sinnxdx\right] \text{ (Since } f(-x) = -f(x)) \\
=& \frac{2}{\pi}\int_0^{\pi} f(x) sinxdx
\end{aligned}
$$

Fourier series for Discontinuous functions: Let the function $f(x)$ be defined by

$$
\begin{aligned}
f(x) =& f_1(x), \quad \alpha < x < x_0 \\
=& f_2(x), \quad x_0 < x < \alpha + 2\pi
\end{aligned}
$$

where x_0 is the point of discontinuity in the interval $(\alpha, \alpha + 2\pi)$

In such cases also, we obtain the Fourier series for $f(x)$ in the usual way. The values of a_0, a_n, b_n are evaluated by

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)dx + \int_{x_0}^{\alpha+2\pi} f_2(x)dx\right]$$
$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)cosnxdx + \int_{x_0}^{\alpha+2\pi} f_2(x)cosnxdx\right]$$
$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)sinnxdx + \int_{x_0}^{\alpha+2\pi} f_2(x)sinnxdx\right]$$

If $x = x_0$ is the point of finite discontinuity, then the sum of the Fourier series

$$= \frac{1}{2}\left[\lim_{h\to 0} f(x_0 - h) + \lim_{h\to 0} f(x_0 + h)\right]$$
$$= \frac{1}{2}\left[f(x_0 - 0) + f(x_0 + 0)\right]$$

THANK YOU