BALASAHEB DESAI COLLEGE, PATAN

"Differential Calculus-I"

MEAN VALUE THEOREMS

By

Dr. D. R. Phadatare
Department of Mathematics,
Balasaheb Desai College, Patan
India (M.S.)
email:phadatared@yahoo.com

Introduction: The function f is defined over an interval $[a, b]$. Let $c \in [a, b]$. If the function f has a special property at c, then c is called mean value and the property is known as mean value theorem. Here we shall consider three mean value theorems namely, Rolle's theorem, Lagrange's theorem and Cauchy's theorem.

Intervals: A number x is said to belong to

(i) a closed interval $[a, b]$ if $a \leq x \leq b$.

(ii) an open interval (a, b) if $a < x < b$.

In the first case x can have all values between a and b including the values, a and b, whereas in the second case the values a and b are excluded.

Rolle's Theorem : If the function $f(x)$

(i) is continuous in the closed interval $[a, b]$,

(ii) is differentiable in the open interval (a, b), and

(iii) $f(a) = f(b)$

then there exists at least value $x = c$ in (a, b) such that $f'(c) = 0$.

Proof: Let $f(x)$ be continuous in $[a, b]$ then it is bounded. Let its upper bound be U and lower bound be L.

If $U = L$, then $f(x) = U = L$, $x \in (a, b)$ and hence $f(x)$ is constant i.e. $f'(x) = 0$.
Now we consider the case when $U \neq L$ then either U or L or both are different from $f(a) = f(b)$. Let U be different from $f(a)$ or $f(b)$, then U attained at least one value of $x = c$ in the open interval (a, b).
$\therefore f(c) = U$. But $U \neq f(a)$ or $f(b)$, so c is different from both a as well as b i.e. $a < c < b$.
When $f'(c)$ is positive we have $f(x) > f(c)$ in the small interval $(c, c + \delta c)$ and when $f'(c)$ is negative we have $f(x) > f(c)$ in the small interval $(c - \delta c, c)$.
But $f(c) = U$. Since, U is the upper bound, $f(x)$ can not be greater than $f(c)$ in $[a, b]$. Hence the only possibility is that $f'(c) = 0$.
Geometrical Interpretation of the Rolle's Theorem : If the graph of $y = f(x)$ be drawn between $x = a$ and $x = b$, then $f(a) = f(b)$ and the curve is continuous. The theorem states that there is at least one point on the curve between the points $x = a$ and $x = b$ at which the tangent to the curve is parallel to the x-axis.

Algebraic Interpretation: If $f(x)$ be polynomial in x, then each term is continuous function of x. Let a and b be the roots of the equation $f(x) = 0$ so that $f(a) = f(b) = 0$. Then Rolle's theorem states that at lest one root of the equation $f'(x) = 0$ lies between a and b.

Ex. 1 Verify Rolle's theorem in the case of function

$$f(x) = 2x^3 + x^2 - 4x - 2$$

Solution: Here $f(x)$ is rational integral function of x, so it is continuous and differentiable for all real values of x. Thus first two conditions of Rolle's theorem are satisfied in any interval.
Now put $f(x) = 0$

$$\begin{aligned} \text{i. e. } 2x^3 + x^2 - 4x - 2 = 0 \\ \Rightarrow (x^2 - 2)(2x + 1) = 0 \\ \Rightarrow x = \pm\sqrt{2}\ , -1/2. \\ \Rightarrow f(\sqrt{2}) = f(-\sqrt{2}) = f(-1/2) = 0\ . \end{aligned}$$

Consider the interval $(-\sqrt{2}, \sqrt{2})$ for which all the conditions of Rolle's theorem are satisfied. We have to see that $f'(x)$ vanishes at least once in the open interval $(-\sqrt{2}, \sqrt{2})$.

$$\begin{aligned} \text{i. e. } f'(x) &= 0 \\ \Rightarrow 6x^2 + 2x - 4 &= 0 \\ \Rightarrow 2(3x-2)(x+1) &= 0 \\ \Rightarrow x &= 2/3, -1 \\ \therefore f(2/3) = f(-1) &= 0 \end{aligned}$$

$\therefore$ The points $x = -1$ and $x = 2/3$ are in the open interval $(-\sqrt{2}, \sqrt{2})$.

Thus the Rolle's theorem is verified.

Ex. 2 Discuss the applicability of Rolle's theorem in case of the following functions.

$$\begin{aligned} (i) f(x) =& |x|, \quad \text{in } [-1, 1] \\ (ii) f(x) =& x^2 + 1, \quad x \in [0, 1] \\ =& 3 - x, \quad x \in [1, 2] \end{aligned}$$

$$(iii) f(x) = tanx, \quad 0 \leq x \leq \pi.$$

Solution: (i) The given function is $f(x) = |x|$, $x \in [-1, 1]$.

$$\therefore f(-1) = 1 \text{ and } f(1) = 1$$
$$\therefore \ f(-1) = f(1).$$

The function $f(x)$ is a continuous at every point in $[-1, 1]$.
Now we have to find $f'(x)$ exist at $x = 0$ or not.

$$Rf'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$\therefore Rf'(x) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| + 0}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$
$$\text{and } Lf'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$
$$\therefore Lf'(x) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{|h| - 0}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$
$$\therefore \ Rf'(0) \neq Lf'(0) \ .$$

$\therefore$ The function $f(x)$ is not differentiable at $x = 0$ in $(-1, 1)$.
$\therefore$ The Rolle's theorem is not applicable to the given function $f(x) = |x|$ in $(-1, 1)$.
(ii) The given function is

$$\begin{aligned} f(x) =& x^2 + 1, \quad x \in [0, 1] \\ =& 3 - x, \quad x \in [1, 2] \\ \therefore \ f(0) =& 1 \text{ and } f(1) = 1 \\ \therefore \ f(0) =& f(1). \end{aligned}$$

The function $f(x)$ is a continuous at every point in $[0, 2]$
Now we have to find $f'(x)$ exist at $x = 0$ or not.

$$Rf'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{3 - (1+h) - (3-1)}{h} =$$

$$= \lim_{h \to 0} \frac{2 - h - 2}{h} = -1$$

$$\text{and } Lf'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$

$$\therefore Lf'(x) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{3-(1-h)}{-h} = \lim_{h\to 0} \frac{2-h}{-h} = 2$$

$\therefore \; Rf'(0) \neq Lf'(0)$.

$\therefore$ The function $f(x)$ is not differentiable at $x = 1$ in $(0, 2)$.
$\therefore$ The Rolle's theorem is not applicable to the given function $f(x)$ in $(0, 2)$.

Ex. 3 Verify Rolle's theorem for the function

$$(i) f(x) = x^2 - 5x + 7 \text{ in } [2, 3]$$
$$(ii) f(x) = sinx \text{ in } [0, \pi]$$
$$(iii) f(x) = x(x+2)e^{-x/2} \text{ in } [-2, 0]$$
$$(iv) f(x) = x^2 \text{ in } [-1, 1].$$

Solution: (i) The given function $f(x) = x^2 - 5x + 7$ is a polynomial so that it is continuous in $[2, 3]$ and differentiable in $(2, 3)$. Also $f(2) = 1$ and $f(3) = 1 \quad \therefore \; f(2) = f(3)$.
Thus all the conditions of Rolle's theorem are satisfied, hence there is a point c in $(2, 3)$ such that $f'(c) = 0$

Now

$$f'(x) = 2x - 5$$
$$\therefore\ f'(c) = 2c - 5 \ \Rightarrow\ c = 5/2$$

Hence $c = 5/2$ lies in the open interval $(2, 3)$.
$\therefore$ Rolle's theorem is verified for the given function $f(x)$.
(ii) The given function $f(x) = sinx$ is obviously continuous in $[0, \pi]$ and differentiable in $(0, \pi)$. Also $f(0) = 0$ and $f(\pi) = 0$
$\therefore f(0) = f(\pi)$.
Thus all the conditions of Rolle's theorem are satisfied, there is a point c in $(0, \pi)$ such that $f'(c) = 0$.
Now

$$f'(x) = cosx \quad \therefore\ f'(c) = cosc$$
$$\therefore\ f'(c) = 0 \Rightarrow\ cosc = 0 \Rightarrow\ c = \pi/2.$$

Hence, $c = \pi/2$ lies in the open interval $(0, \pi)$.
$\therefore$ Rolle's theorem is verified for the given function $f(x)$.

Lagrange's Mean Value Theorem : If the function $f(x)$
(i) is continuous in the closed interval $[a, b]$,
(ii) is differential in the open interval (a, b),
then there exists a value c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof: Consider the function $\phi(x) = f(x) + Ax$, where A is such that $\phi(a) = \phi(b)$.
Thus

$$\begin{aligned} \phi(a) =& f(a) + Aa = f(b) + Ab = \phi(b) \\ \Rightarrow\ & A = -\frac{f(b) - f(a)}{b - a} \end{aligned} \tag{1}$$

Here $\phi(x)$ is continuous in $[a, b]$ and differentiable in (a, b). Since $\phi(x)$ satisfies all the conditions of Rolle's theorem. There exists c, where $a < c < b$ such that $\phi'(c) = 0$.

Now

$$\phi'(x) = f'(x) + A$$
$$\text{Hence } \phi'(c) = 0 \Rightarrow \ f'(c) + A = 0$$
$$\Rightarrow f'(c) = -A$$
$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a} \quad [\because \text{by } (1)]$$

Geometrical Interpretation of L. M. V. T.: Let on the graph $y = f(x)$, A and B are the points corresponding to $x = a$ and $x = b$.

$$\text{Slope of the chord AB} = \frac{QB}{QA}$$
$$= \frac{f(b) - f(a)}{b - a}.$$

Slope of the tangent to the curve at P where the point P on the curve corresponds to $x = c$ is $f'(c)$. Hence slope of the chord is equal to the slope of the tangent.

Thus geometrically the theorem states that if a curve is continuous in $[a, b]$, has tangent at every point (a, b) then there is a point c between A and B such that the tangent at c is parallel to the chord AB.

Ex. 1 Verify Lagrange's Mean Value theorem for

$$\begin{aligned}(i) f(x) =& x(x-1)(x-2) \quad \text{in } [0, 1/2] \\ (ii) f(x) =& x^3 + 3x^2 - 5x \quad \text{in } [1, 2] \\ (iii) f(x) =& e^x \quad \text{in } [0, 1]\end{aligned}$$

Solution: (i) We have

$$f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x \tag{1}$$

Here $f(x)$ is a polynomial, it must be continuous in $[0, 1/2]$ and differentiable in $(0, 1/2)$. Thus function satisfies the conditions of L. M. V. T.

Hence,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{2}$$

$$f'(x) = 3x^2 - 6x + 2 \quad [\because \text{ by } (1)]$$
$$a = 0, \ f(a) = f(0) = 0;$$
$$b = 1/2, \ f(b) = f(1/2) = 3/8$$
$$f'(c) = 3c^2 - 6c + 2$$
$$\therefore 3c^2 - 6c + 2 = \frac{3/8 - 0}{1/2 - 0} = 3/4 \ \ [\because \text{ by } (2)]$$
$$\therefore 3c^2 - 6c + 5/4 = 0 \Rightarrow c = 1 \pm \frac{\sqrt{21}}{6}$$

$\therefore c = 1 + \frac{\sqrt{21}}{6} \notin (0, 1/2)$ and $c = 1 - \frac{\sqrt{21}}{6} \in (0, 1/2)$
$\therefore$ L. M. V. is verified for the given function $f(x)$.
Ex. 2 Apply L. M. V. T. to show that

$$(i)\ 0 < \frac{1}{log(1+x)} - \frac{1}{x} < 1$$
$$(ii)\ 1 < \frac{sin^{-1}x}{x} < \frac{1}{\sqrt{(1-x^2)}} \text{ for } 0 \leq x < 1.$$

(i) Consider $f(x) = log(1 + x)$ and apply L. M. V. T. on $[0, x]$.
Since

$$\begin{aligned} f'(c) &= \frac{f(x) - f(0)}{x - 0} \\ \therefore \ \frac{1}{1+c} &= \frac{log(1+x) - 0}{x - 0} \qquad [\because \ f'(x) = \frac{1}{1+x}] \\ \therefore \ \frac{1}{1+c} &= \frac{log(1+x)}{x}, \ \text{for } 0 < c < x \qquad (1) \end{aligned}$$

As $0 < c < x$

$$\begin{aligned} \therefore \ 1 < 1 + c < 1 + x \\ \text{Hence }, 1 < \frac{x}{log(1+x)} < 1 + x \ \ [\because \ \text{by } (1)] \\ \therefore \ \frac{1}{x} < \frac{1}{log(1+x)} < \frac{1+x}{x} \\ \therefore \ \frac{1}{x} - \frac{1}{x} < \frac{1}{log(1+x)} - \frac{1}{x} < \frac{1}{x} + 1 - \frac{1}{x} \end{aligned}$$

$$\therefore \; 0 < \frac{1}{log(1+x)} - \frac{1}{x} < 1$$

Hence the proof.

Cauchy's Mean value Theorem: If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and derivable in (a, b), then there exist a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \; , g(x) \neq 0.$$

Proof: Consider the function, $\phi(x) = f(x) + Ag(x)$.
Where A is a constant such that $\phi(a) = \phi(b)$.

$$\therefore \; \phi(a) = f(a) + Ag(a) = f(b) + Ag(b) = \phi(b)$$

$$\Rightarrow \; A = -\frac{f(b) - f(a)}{g(b) - g(a)} \; , \text{where } g(b) \neq g(a) \qquad (1)$$

If $g(b) = g(a)$, then by Rolle's theorem $g'(x) = 0$, for some x in the interval $a < x < b$ which is against hypothesis.

Again, $\phi(x)$ is continuous in $[a, b]$ and derivable in (a, b) and $\phi(a) = \phi(b)$. Then by Rolle's theorem $\phi'(c) = 0$.

$$\begin{aligned} \text{But } \ \phi'(x) &= f'(x) + Ag'(x) \\ \Rightarrow \phi'(c) &= f'(c) + Ag'(c) = 0 \\ \Rightarrow A &= -\frac{f'(c)}{g'(c)} \\ \Rightarrow \frac{f'(c)}{g'(c)} &= \frac{f(b) - f(a)}{g(b) - g(a)} \quad [\because \text{ by } (1)] \end{aligned}$$

Hence the proof.

Ex. 1 Verify Cauchy Mean value theorem for the function defined below.

$$\begin{aligned} (i) f(x) =& \frac{1}{x}, \ g(x) = \frac{1}{x^2}, \quad \text{on } [1, 4] \\ (ii) f(x) =& 3x + 2, \ g(x) = x^2 + 1, \quad \text{on } [1, 4] \\ (iii) f(x) =& e^x, \ g(x) = e^{-x}, \quad \text{on } [0, 1] \\ (iv) f(x) =& sinx, \ g(x) = cosx, \quad \text{on } [-\frac{\pi}{2}, 0] \end{aligned}$$

Solution: (i) Since, $f(x)$, $g(x)$ are continuous in $[1, 4]$ and derivable in $(1, 4)$.
Given

$$f(x) = \frac{1}{x},\ g(x) = \frac{1}{x^2}$$
$$\therefore\ f'(x) = -\frac{1}{x^2},\ g'(x) = -\frac{2}{x^3}$$
$$\text{and } f(a) = f(1) = 1,\ f(b) = f(4) = \frac{1}{4},$$
$$\text{and } g(a) = g(1) = 1,\ g(b) = g(4) = \frac{1}{16}.$$

$\therefore$ By C.M.V.T. there exists $c \in (1, 4)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$
$$\therefore\ \frac{c}{2} = \frac{4}{5}$$
$$\therefore\ c = \frac{8}{5} = 1.6 \in (1, 4)$$

$\therefore$ C.M.V.T. is verified.

(iii) Since, $f(x)$, $g(x)$ are continuous in $[-\frac{\pi}{2}, 0]$ and derivable in $(-\frac{\pi}{2}, 0)$.
We have

$$\begin{aligned} f(x) =& sinx,\ g(x) = cosx \\ \therefore\ f'(x) =& cosx,\ g'(x) = -sinx \\ \text{and } f(a) =& f(-\frac{\pi}{2}) = -1,\ f(b) = f(0) = 0, \\ \text{and } g(a) =& g(-\frac{\pi}{2}) = 0,\ g(b) = g(0) = 1. \end{aligned}$$

$\therefore$ By C.M.V.T. there exists $c \in (-\frac{\pi}{2}, 0)$ such that

$$\begin{aligned} \frac{f'(c)}{g'(c)} =& \frac{f(b) - f(a)}{g(b) - g(a)} \\ \therefore\ \frac{cosc}{sinc} =& \frac{-1}{1} \\ \therefore\ cotc =& -1 \Rightarrow c = cot^{-1}(-1) = -\frac{\pi}{4} \in (-\frac{\pi}{2}, 0) \end{aligned}$$

$\therefore$ C.M.V.T. is verified.

Ex. 2 If $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ then show that c is geometric mean between and a and b where $a, b > 0$.
Solution: Since $f(x)$, $g(x)$ are continues in $[a, b]$ and derivable in (a, b).
Now $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$

$$\therefore\ f'(x) = \frac{1}{2}x^{-1/2},\ g'(x) = -\frac{1}{2}x^{-3/2}$$
$$\text{and } f(a) = a^{1/2},\ f(b) = b^{1/2},\ g(a) = a^{-1/2},\ g(b) = b^{-1/2}.$$

$\therefore$ By C.M.V.T. there exists $c \in (a, b)$such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$
$$\therefore\ c = \sqrt{ab} \in (a, b)$$

Hence C. M. V. T. is verified and c is a geometric mean between a and b.

Ex. 3 Show that :

$$(i)\ \frac{sinb - sina}{e^b - e^a} = \frac{cosc}{e^c},\ a < c < b$$

$$(ii)\ \frac{sinb - sina}{cosb - cosa} = -cotc,\ a < c < b.$$

Solution: (i) Let $f(x) = sinx$, $g(x) = e^x$

$$\therefore\ f'(x) = cosx,\ g'(x) = e^x$$

and $f(a) = sina,\ f(b) = sinb,\ g(a) = e^a,\ g(b) = e^b$

$\therefore$ By C.M.V.T. there exists $c \in (a, b)$such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\therefore\ \frac{cosc}{e^c} = \frac{sinb - sina}{e^b - e^a},\ a < c < b$$

Hence the proof.

Taylor's Theorem: If (i) $f(x)$ and its first $(n-1)$ derivatives be continuous in $[a, a+h]$, and (ii) $f^n(x)$ exists for every value of x in $(a, a+h)$, then there is at least one number θ $(0 < \theta < 1)$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^n}{n!}f^n(a+\theta h) \qquad (1)$$

which is called Taylor's theorem with Lagrange's form of remainder, the remainder R_n being $\frac{h^n}{n!}f^n(a+\theta h)$.

Cor. 1 taking $n = 1$ in (1), Taylor's theorem reduces to Lagrange's mean value theorem.

Cor.2 Putting $a = 0$ and $h = x$ in (1), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^n}{n!}f^n(\theta x). \qquad (2)$$

which is known as Maclaurin's theorem with Lagrange's form of remainder, the remainder R_n being $\frac{x^n}{n!}f^n(\theta x)$.

Expansions of functions:

(1) **Maclaurin's Series:** If $f(x)$ can be expanded as an infinite series, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \infty \tag{3}$$

If $f(x)$ possesses derivatives of all orders and the remainder R_n in (2) tends to zero as $n \to \infty$, then the Maclaurin's theorem becomes the Maclaurin's series (3).

(2) **Taylor's Series:** If $f(x+h)$ can be expanded as an infinite series, then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + ... + \infty \tag{4}$$

If $f(x)$ possesses derivatives of all orders and the remainder R_n in (1) tends to zero as $n \to \infty$, then the Taylor's theorem becomes the Taylor's series (4).

Cor. Replacing x by a and h by $(x-a)$ in (4), we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + ... + \infty$$

Taking $a = 0$, we get Maclaurin's series.
Ex. 1 Find the series expansion of e^{ax}.
Solution: let

$$\begin{aligned} f(x) &= e^{ax} & \therefore\ f(0) &= 1 \\ f'(x) &= ae^{ax} & \therefore\ f'(0) &= a \\ f''(x) &= a^2 e^{ax} & \therefore\ f(0) &= a^2 \\ f'''(x) &= a^3 e^{ax} & \therefore\ f(0) &= a^3 \end{aligned}$$

and so on.
Now, substituting the values of $f(0), f'(0), f''(0)$ etc. in

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + ... \qquad \text{we get} \\ e^{ax} &= 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + ... \end{aligned}$$

which is required expansion of e^{ax} in ascending powers of x.
Cor. When $a = 1$, we get $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

Ex. 2 Expand *sinx* in ascending powers of x.
Solution:Let

$$\begin{aligned} f(x) &= sinx & \therefore\ f(0) &= 0 \\ f'(x) &= cosx & \therefore\ f'(0) &= 1 \\ f''(x) &= -\ sinx & \therefore\ f(0) &= 0 \\ f'''(x) &= -\ cosx & \therefore\ f(0) &= -1 \\ f^{iv}(x) &= sinx & \therefore\ f(0) &= 0 \\ f^{v}(x) &= cosx & \therefore\ f^{v}(0) &= 1 \end{aligned}$$

and so on.
Now, substituting the values of $f(0), f'(0), f''(0)$ etc. in

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... \qquad \text{we get}$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

which is required expansion of *sinx* in ascending powers of x.

Ex. 3 Expand $(1+x)^n$ in ascending power of x.
Solution:Let

$$\begin{aligned} f(x) =& (1+x)^n & \therefore\ f(0) = 1 \\ f'(x) =& n(1+x)^{n-1} & \therefore\ f'(0) = n \\ f''(x) =& n(n-1)(1+x)^{n-2} & \therefore\ f(0) = n(n-1) \end{aligned}$$

and so on.
Now, substituting the values of $f(0), f'(0), f''(0)$ etc. in

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... \qquad \text{we get}$$

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + ...$$

which is required expansion of $(1+x)^n$ in ascending powers of x.
Note: When $n = -1$, we get,

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + ...$$

Putting x by $-x$, we get

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + ...$$

Ex. 4 Expand $log(1+x)$ in ascending powers of x.
Solution:Let

$$\begin{aligned} f(x) =& log(1+x) & \therefore \ f(0) = 0 \\ f'(x) =& \frac{1}{1+x} & \therefore \ f'(0) = 1 \\ f''(x) =& -(1+x)^{-2} & \therefore \ f(0) = -1 \\ f'''(x) =& 2(1+x)^{-3} & \therefore \ f(0) = 2 \end{aligned}$$

and so on.
Now, substituting the values of $f(0), f'(0), f''(0)$ etc. in

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... \qquad \text{we get}$$

$$log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$$

which is required expansion of $log(1+x)$ in ascending powers of x.
Note: Replacing x by $-x$ we get,

$$log(1-x) = -[x + \frac{x^2}{2} + \frac{x^3}{3} + ...]$$

Indeterminate Forms: The forms $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, \infty^0, 1^\infty$ are known as indeterminate forms. In chapter on Limits, we found out the limits of some of the functions which assume the indeterminate form $\frac{0}{0}$ for a particular value of the variable, for example, the function $\frac{sinx}{x}$ assumes the indeterminate form $\frac{0}{0}$ when $x = 0$, but $\lim_{x \to 0} \frac{sinx}{x} = 1$.

Below will be the given methods of evaluating the limits of the indeterminate forms through the use of differentiation and expansion in series.

Indeterminate Form $\frac{0}{0}$ (L, Hospitals Rule) : If $f(a) = 0 = g(a)$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Cor. 1 If $f'(a), f''(a), ...f^{n-1}(a)$ and $g'(a), g''(a), ...g^{n-1}(a)$ are all zero, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^n(x)}{g^n(x)}$.

Cor. 2 If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

The Indeterminate Form $\frac{\infty}{\infty}$: If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. The same rule as that for evaluating the indeterminate form $\frac{0}{0}$.

Cor. 1: The form $0 \times \infty$:If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then the product $f(x) \times g(x)$ is the form $0 \times \infty$. It may be transformed into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by one of the relations.

$$f(x) \times g(x) = \frac{f(x)}{\frac{1}{g(x)}} = \frac{g(x)}{\frac{1}{f(x)}}$$

Cor. 2: The other indeterminate forms can be reduced to the form $\lim_{x\to a} f(x) = 0$ or to the form $\lim_{x\to a} g(x) = \infty$. If the base and the index are both functions of x, say $[f(x)]^{g(x)}$ then we can write $[f(x)]^{g(x)} = e^{g(x) log f(x)}$ and evaluate the limit $g(x) \times log f(x)$ by previous rules.

Ex. 1 Evaluate the limit, $\lim_{x\to 0} \frac{e^x + sinx - 1}{log(1+x)}$

Solution: This is of the form $\frac{0}{0}$

Here $f(x) = e^x + sinx - 1$ and $g(x) = log(1 + x)$

$$\therefore \ f'(x) = e^x + cosx, \quad g'(x) = \frac{1}{1+x}$$
$$\therefore \ f'(0) = 2, \quad g'(0) = 1$$

Hence, by rule,

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = 2.$$

Ex. 2 Find $\lim_{x \to 0} \frac{2cosx - 2 + x^2}{x^4}$

Solution: This is of the form $\frac{0}{0}$

Let $f(x) = 2cosx - 2 + x^2$ and $g(x) = x^4$

$$\therefore \ f'(x) = -2sinx + 2x, \quad g'(x) = 4x^3$$
$$\therefore \ f''(x) = -2cosx + 2, \quad g''(x) = 12x^2$$
$$\therefore \ f'''(x) = 2sinx, \quad g'''(x) = 24x$$
$$\therefore \ f^{iv}(x) = 2cosx, \quad g^{iv}(x) = 24$$

Hence, by rule,

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{f^{iv}(0)}{g^{iv}(0)}=\frac{2}{24}=\frac{1}{12}.$$

Ex. 3 Evaluate $\lim_{x\to 1}[\frac{x}{x-1}-\frac{1}{logx}]$

Solution: This is the form $\infty-\infty$

$$\therefore\ \lim_{x\to 1}\Big[\frac{x}{x-1}-\frac{1}{logx}\Big]=\lim_{x\to 1}\frac{xlogx-x+1}{(x-1)logx}\ \text{ which is the form }\frac{0}{0}$$

let $f(x)=xlogx-x+1, \quad g(x)=(x-1)logx$

$$f'(x)=1+logx-1, \quad g'(x)=1-\frac{1}{x}+logx$$

$$f''(x)=\frac{1}{x}, \quad g''(x)=\frac{1}{x^2}+\frac{1}{x}$$

Hence, by rule,

$$\lim_{x\to 1}\frac{f(x)}{g(x)}=\frac{f'''(1)}{g'''(1)}=\frac{1}{2}.$$

Ex. 4 Evaluate $\lim_{x\to 0} xlogsinx$

Solution:This is in the form $0 \times (-\infty)$

$$\therefore \lim_{x\to 0} xlogsinx = \lim_{x\to 0} \frac{logsinx}{\frac{1}{x}} \text{ which is the form } \frac{\infty}{\infty}$$

$$\text{let } f(x) = logsinx, \quad g(x) = \frac{1}{x}$$

$$f'(x) = cotx, \quad g'(x) = -\frac{1}{x^2}$$

$$f''(x) = -cosec^2 x, \quad g''(x) = -\frac{2}{x^3}$$

Hence, by rule,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^2}{sin^2 x} \cdot x = 1 \cdot 0 = 0.$$

Ex. 5 Evaluate $\lim_{x\to 0} (1 + sinx)^{cotx}$

Solution: Let

$$\lim_{x\to 0} (1 + sinx)^{cotx} = \lim_{x\to 0} e^{log(1+sinx)^{cotx}} = \lim_{x\to 0} e^{cotxlog(1+sinx)} \quad (1)$$

Let us find $\lim_{x\to 0} cotxlog(1+sinx)$. This is of the form $\infty \times 0$.

$$\therefore \lim_{x\to 0} cotxlog(1+sinx) = \lim_{x\to 0} \frac{log(1+sinx)}{tanx}$$
$$= \lim_{x\to 0} \frac{log(1+sinx)}{x} \cdot \frac{x}{tanx} \quad \text{[Note this step]}$$
$$= \lim_{x\to 0} \frac{log(1+sinx)}{x} \lim_{x\to 0} \frac{x}{tanx}$$
$$= \lim_{x\to 0} \frac{log(1+sinx)}{x} \cdot 1$$
$$= \lim_{x\to 0} \frac{log(1+sinx)}{x} \quad \text{This is of the form } \frac{0}{0}$$

Let $f(x) = log(1+sinx)$ and $g(x) = x$

$$\therefore f'(x) = \frac{cosx}{1+sinx}, \quad g'(x) = 1$$
$$\therefore \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(0)}{g'(0)} = \frac{1}{1} = 1$$

Hence, from equation (1), $\lim_{x\to 0} cotxlog(1+sinx) = \lim_{x\to 0} e^1 = e$.

Ex. 6 Find the values of a, b, c so that $\lim_{x \to 0} \frac{ae^x - bcosx + ce^{-x}}{xsinx} = 2$

Solution:

$$\text{Let } \frac{ae^x - bcosx + ce^{-x}}{xsinx} = \frac{f(x)}{g(x)} \text{ so that } \frac{f(0)}{g(0)} = \frac{0}{0},$$

$$\therefore a - b + c = 0 \tag{1}$$

$$\frac{f'(x)}{g'(x)} = \frac{ae^x + bsinx - ce^{-x}}{xcosx + sinx}; \ \frac{f'(0)}{g'(0)} = \frac{0}{0}$$

$$\therefore a - c = 0 \tag{2}$$

$$\frac{f''(x)}{g''(x)} = \frac{ae^x + bcosx + ce^{-x}}{2cosx - xsinx}; \ \frac{f''(0)}{g''(0)} = \frac{a+b+c}{2}$$

$$\therefore a + b + c = 4 \quad [\because \lim_{x \to 0} \frac{f''(x)}{g''(x)} = 2] \tag{3}$$

Solving equations (1), (2) and (3), we get

$$a = 1, \ b = 2, \ c = 1.$$

THANK U