Koyana Education Society's
BALASAHEB DESAI COLLEGE, PATAN

"SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS AND SIMULTANEOUS DIFFERENTIAL EQUATIONS"

By
Mr. D. R. Phadatare
Department of Mathematics,
Balasaheb Desai College, Patan
India (M.S.)
email:phadatared@yahoo.com

1.1.1 The General form:

The general (standard) form of the linear differential of the second order such equations are of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \ , \qquad (1)$$

where P, Q and R are functions of x only.
The equation (1) can be written in symbolic form as $f(D)y = R$ where $D = \frac{d}{dx}$. In this chapter we shall discuss the following types, of solving the equation (1) when

1. one solution of $f(D)y = 0$ is known (General Method of solution)
2. Solution by the Change of dependent variable (Removal of first order derivative)
3. Solution by the change of independent variable
4. Method of variation of parameter

1.1.2 Type I: Complete Solution when one integral is known (General Method of solution)

Consider the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \ . \qquad (1)$$

If $y = u$ is given solution belonging to the complementary function of the equation (1).

$$\therefore \quad \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \ . \qquad (2)$$

If the other solution be $y = v$, then

$$y = uv \ , \qquad (3)$$

be complete solution of the differential equation (1). Now differentiating equation (3) with respect to x we get

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \ , \qquad (4)$$

and

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2uv\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} \ . \qquad (5)$$

Using equations (3), (4) and (5) in equation (1), we get

$$\left(u\frac{d^2v}{dx^2} + 2uv\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}\right) + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R \ ,$$

i. e.

$$u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R \ . \qquad (6)$$

Now using equation (2), the equation (6) becomes

$$u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} = R \ ,$$

dividing by u, we get

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \ . \qquad (7)$$

On putting $\frac{dv}{dx} = t$, equation (7) becomes

$$\frac{dt}{dx} + \left(P + \frac{2}{u}\frac{du}{dx}\right)t = \frac{R}{u} \ .$$

Which is linear equation hence its solution is given by

$$t \cdot \text{I. F.} \ = \int \text{I. F.} \ Qdx + c \ , \qquad (8)$$

where

$$\text{I. F.} = e^{\int \left(P + \frac{2}{u}\frac{du}{dx}\right)} = e^{\int Pdx} e^{2\int \frac{du}{u}} \ ,$$

$$= e^{\int Pdx} e^{2logu} = e^{\int Pdx} e^{logu^2} = u^2 e^{\int Pdx} \ .$$

Hence the equation (8) becomes

$$t \cdot u^2 e^{\int Pdx} = \int \frac{R}{u} u^2 e^{\int Pdx} dx + c \ ,$$

dividing by $u^2 e^{\int Pdx}$, and re substituting the value of t, we get

$$\frac{dv}{dx} = \frac{e^{-\int Pdx}}{u^2} \int R \cdot u e^{\int Pdx} dx + c \frac{e^{-\int Pdx}}{u^2} ,$$

integrating we get

$$v = \int \left(\frac{e^{-\int Pdx}}{u^2} \int R \cdot u e^{\int Pdx} dx + c \frac{e^{-\int Pdx}}{u^2} \right) dx + c_1 ,$$

Hence, from equation (3) complete solution of equation (1) is given by

$$y = uv = u \left[\int \left(\frac{e^{-\int Pdx}}{u^2} \int R \cdot u e^{\int Pdx} dx + c \frac{e^{-\int Pdx}}{u^2} \right) dx + c_1 \right]$$

To find out the integral belonging to the complementary function we follow the following rules:

1. $y = x$ is a known integral if $P + xQ = 0$
2. $y = e^x$ is a known integral if $1 + P + Q = 0$
3. $y = e^{-x}$ is a known integral if $1 - P + Q = 0$
4. $y = e^{mx}$ is a known integral if $m^2 + mP + Q = 0$
5. $y = x^2$ is a known integral if $2 + 2Px + Qx^2 = 0$
6. $y = x^n$ is a known integral if $n(n-1) + nPx + Qx^2 = 0$

Example 1.1:

Solve

$$x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0$$

Solution: Dividing by x

$$\frac{d^2y}{dx^2} - (2-\frac{1}{x})\frac{dy}{dx} + (1-\frac{1}{x})y = 0 \ , \qquad (1)$$

$$\therefore P = -2 + \frac{1}{x}, Q = 1 - \frac{1}{x}, R = 0$$

$$\therefore 1 + P + Q = 1 - 2 + \frac{1}{x} + 1 - \frac{1}{x} = 0$$

$\therefore y = e^x$ is known solution of equation (1). Let the general solution be

$$y = uv = e^x v \ . \qquad (2)$$

Using equation (2), equation (1) transforms to

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \ .$$

$$\therefore \frac{d^2v}{dx^2} + (\frac{1}{x} - 2 + \frac{2}{e^x}e^x)\frac{dv}{dx} = 0 \ ,$$

$$\therefore \frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} = 0 \ , \qquad (3)$$

put

$$\frac{dv}{dx} = t \ , \quad \therefore \frac{d^2v}{dx^2} = \frac{dt}{dx} \ , \qquad (4)$$

using equation (4) in equation (3), we get

$$\frac{dt}{dx} + \frac{1}{x}t = 0$$

integrating we get

$$logt + logx = logc_1 \ , \Rightarrow tx = c_1 \ ,$$

$$\Rightarrow \frac{dv}{dx}x = c_1 \Rightarrow v = c_1 logx + c_2$$

Hence from equation (2) complete solution is given by

$$y = e^{x}(c_1 logx + c_2) \ .$$

1.1.3 Type II: Transformation of the equation by changing the dependent variable. (Removal of First order Derivative)

In this type, we change the dependent variable y to v such that the coefficient of its first order derivative is zero, so that, the resulting differential equation can be easily integrable.
Consider the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \ . \tag{1}$$

put

$$y = uv \ , \tag{2}$$

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \ ,$$

$$\therefore \quad \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2uv\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} \ .$$

On putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1) we get

$$\left(u\frac{d^2v}{dx^2}+2uv\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}\right)+P\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)+Quv=R\ ,$$

$$\Rightarrow u\frac{d^2v}{dx^2}+\left(Pu+2\frac{du}{dx}\right)\frac{dv}{dx}+\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)v=R\ ,$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}+\frac{1}{u}\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)v=\frac{R}{u}\ . \tag{3}$$

Here we shall remove the first derivative.

$$P+\frac{2}{u}\frac{du}{dx}=0\ , \tag{4}$$

$$\Rightarrow \frac{du}{u}=-\frac{1}{2}Pdx\ ,$$

integrating we get

$$logu=-\frac{1}{2}\int Pdx\ ,$$

$$\therefore\quad u=e^{-\frac{1}{2}\int Pdx}\ . \tag{5}$$

Now from equation (4) we have

$$\frac{du}{dx} = -\frac{1}{2}Pu ,$$

$$\Rightarrow \frac{d^2u}{dx^2} = \frac{1}{4}P^2u - \frac{1}{2}u\frac{dP}{dx} . \qquad (6)$$

Using the equation (6), the last term in the L. H. S. of the equation (3) becomes

$$\frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} ,$$

Hence the equation (3) transformed to

$$\frac{d^2v}{dx^2} + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right)v = Re^{\frac{1}{2}\int Pdx} ,$$

$$\Rightarrow \frac{d^2v}{dx^2} + Q_1v = R_1 , \qquad (7)$$

where

$$Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} ,$$

$$R_1 = Re^{\frac{1}{2}\int Pdx} .$$

Since equation (7) is easily integrable, find v and substituting the values of u and v thus obtained in the equation (2) we get the complete solution of the equation (1).

$$y = uv \quad \text{and} \quad u = e^{-\frac{1}{2}\int P dx}$$

Example 1.9: Solve

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = x^3e^x$$

Solution: Dividing by x^2

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + (1+\frac{2}{x^2})y = xe^x \ , \qquad (1)$$

$$\therefore \quad P = -\frac{2}{x}, Q = 1+\frac{2}{x^2}, R = xe^x \quad \text{and} \ ,$$

$$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -\frac{2}{x}dx} = e^{\int \frac{dx}{x}} = e^{logx} = x \ .$$

put

$$y = uv = xv \ . \qquad (2)$$

Using the equation (2), the equation (1) transforms to

$$\frac{d^2v}{dx^2} + Q_1 v = R_1 \ , \tag{3}$$

where

$$\begin{aligned} Q_1 =& Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} \ , \\ =& 1 + \frac{2}{x^2} - \frac{1}{4}(\frac{4}{x^2}) - \frac{1}{2}(\frac{2}{x^2}) \ , \\ =& 1 \end{aligned}$$

and

$$\begin{aligned} R_1 =& Re^{\frac{1}{2}\int P dx} \ , \\ =& xe^x e^{-\frac{1}{2}\int -\frac{2}{x} dx} \ , \\ =& xe^x e^{-logx} = xe^x e^{logx^{-1}} = xe^x x^{-1} = e^x \ . \end{aligned}$$

Substituting the values of Q_1 and R_1 in equation (3) we get

$$\frac{d^2v}{dx^2} + v = e^x \Rightarrow \ (D^2+1)v = e^x \ ,$$

$$\therefore \quad \text{A. E. is } \ D^2+1=0 \Rightarrow \ D = \pm i \ ,$$

$$\therefore \quad \text{C. F. } \ = c_1 cosx + c_2 sinx \ ,$$

$$\text{and } \ \text{P. I. } \ = \frac{1}{D^2+1} e^x = \frac{1}{2} e^x \ ,$$

$$\therefore \quad v = c_1 cosx + c_2 sinx + \frac{1}{2} e^x \ .$$

Hence from equation (2) we get complete solution as

$$y = xv = x[c_1 cosx + c_2 sinx] + \frac{x}{2} e^x \ .$$

1.1.4 Type III: Transformation of the Equation by changing the independent variable

Consider the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \ . \tag{1}$$

Let us change the independent variable x to z and $z = f(x)$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

Now substituting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1) we get

$$\left[\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}\right] + P\left(\frac{dy}{dz} \cdot \frac{dz}{dx}\right) + Qy = R\ ,$$

$$\Rightarrow \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \left[\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right]\frac{dy}{dz} + Qy = R\ ,$$

$$\Rightarrow \frac{d^2y}{dz^2} + \frac{\left[\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right]}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}y = \frac{R}{\left(\frac{dz}{dx}\right)^2}\ ,$$

$$\Rightarrow \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1\ , \qquad (2)$$

where

$$P_1 = \frac{\left[\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right]}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} .$$

Here equation (2) can be solved either by taking $P_1 = 0$ **[First Method]** or $Q_1 =$ a constant **[Second Method]**

Procedure:

1. Write the equation with coefficient of $\frac{d^2y}{dx^2}$ as unity if it is not so.
2. Compare the given equation with $y'' + Py' + Qy = R$ and find the values of P, Q and R.
3. Find the values of P_1, Q_1 and R_1 using the formulae
$$P_1 = \frac{\left[\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right]}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} .$$
4. Find the values of z by taking $P_1 = 0$ **[First Method]** or $Q_1 =$ a constant **[Second Method]**

1. We get a reduced equation $\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$ on solving this equation we can find the value of y in terms of z. Then write down the solution in terms of x by replacing the value of z.

Example 1.16: Solve

$$\frac{d^2y}{dx^2} - cotx \frac{dy}{dx} + sin^2 x \cdot y = 0$$

Solution: We have

$$P = -cotx, Q = sin^2 x, R = 0$$

Changing the independent variable from x to z, given equation becomes

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \ , \tag{1}$$

where

$$P_1 = \frac{\left[\frac{d^2z}{dx^2} + P \frac{dz}{dx}\right]}{\left(\frac{dz}{dx}\right)^2}, Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} \ , \tag{2}$$

First Method:

Let us take $P_1 = 0$

$$\therefore P_1 = 0 \Rightarrow \frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0 \Rightarrow \frac{d^2z}{dx^2} - cotx\frac{dz}{dx} = 0 \ , \qquad (3)$$

put

$$\frac{dz}{dx} = t \quad \therefore \frac{d^2z}{dx^2} = \frac{dt}{dx} \ ,$$

hence, equation (3) becomes

$$\frac{dt}{dx} - cotx \cdot t = 0 \Rightarrow \frac{dt}{t} = cotxdx \Rightarrow logt = logsinx + logc \Rightarrow t = c \cdot sinx$$

$$\therefore \quad \frac{dz}{dx} = c \cdot sinx \Rightarrow dz = c \cdot sinxdx \Rightarrow z = -c \cdot cosx \ ,$$

hence from (2) we get

$$Q_1 = \frac{sin^2x}{c^2sin^2x} = \frac{1}{c^2} \ , R_1 = 0 \ ,$$

Now equation (1) becomes

$$\frac{d^2y}{dz^2} + \frac{1}{c^2}y = 0 \Rightarrow (D^2 + \frac{1}{c^2})y = 0 \Rightarrow D^2 + \frac{1}{c^2} \Rightarrow D =$$

$$\therefore \quad \text{C. F.} = c_1 cos\frac{z}{c} + c_2 sin\frac{z}{c} \Rightarrow \text{C. F.} = c_1 cos\left(\frac{-ccosx}{c}\right) + c_2 sin\left(\frac{-ccc}{c}\right.$$

$$\therefore \quad \text{C. F.} = Acos(cosx) + Bsin(cosx) \Rightarrow y = Acos(cosx) + Bsin($$

Second Method: Let us take $Q_1 = \text{Constant}$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{1}{c^2} \Rightarrow \left(\frac{dz}{dx}\right)^2 = c^2 Q \Rightarrow \left(\frac{dz}{dx}\right)^2 = c^2 sin^2 x \ ,$$

$$\therefore \quad \frac{dz}{dx} = c \cdot sinx \Rightarrow dz = c \cdot sinxdx \Rightarrow z = -c \cdot cosx \ ,$$

hence from (2) we get

$$P_1 = \frac{\left[\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right]}{\left(\frac{dz}{dx}\right)^2} \ , \Rightarrow P_1 = \frac{[c \cdot cosx - cotx \cdot csinx]}{c^2 sin^2 x} \ , \Rightarrow P_1 = 0 \ ,$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$$

putting values of P_1, Q_1 and R_1 in (1) we get

$$\frac{d^2y}{dz^2} + \frac{1}{c^2}y = 0 \Rightarrow (D^2 + \frac{1}{c^2})y = 0 \Rightarrow D^2 + \frac{1}{c^2} \Rightarrow D = \pm\frac{i}{c} \ ,$$

$$\therefore \quad \text{C. F.} = c_1 cos\frac{z}{c} + c_2 sin\frac{z}{c}$$

$$\Rightarrow \text{C. F.} = c_1 cos\left(\frac{-ccosx}{c}\right) + c_2 sin\left(\frac{-ccosx}{c}\right) \ ,$$

$$\therefore \quad \text{C. F.} = Acos(cosx) + Bsin(cosx) \ ,$$

$$\therefore \quad y = Acos(cosx) + Bsin(cosx) \ .$$

1.1.5 Type IV: Method of Variation of Parameters

Consider the differential equation

$$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = R \ . \qquad (1)$$

To find particular integral of (1), let complementary function

$$\text{C. F.} = c_1 u + c_2 v \ ,$$

where u and v are functions of x satisfies the equation

$$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = 0 \ . \tag{2}$$

Let us assume particular integral as

$$\text{P. I.}(y) = Xu + Yv \ , \tag{3}$$

where X and Y are unknown function of x. Differentiating (3) with respect to x we get $y' = uX' + Xu' + vY' + Yv'$ assuming that X, Y satisfies the equation

$$X'u + Y'v = 0 \ , \tag{4}$$

then

$$y' = Xu' + Yv' \ . \tag{5}$$

Now differentiating (5) with respect to x we get
$y'' = Xu'' + X'u' + Yv'' + Y'v'$.
Substituting the values of y, y' and y'' in the equation (1), we get

$$A(Xu'' + X'u' + Yv'' + Y'v') + B(Xu' + Yv') + C(Xu + Yv) = R \ ,$$
$$\Rightarrow X(Au'' + Bu' + Cu) + Y(Av'' + Bv' + Cv) + A(X'u' + Y'v') = R \ , \quad (6)$$

where u and v will satisfies equation (1).

$$\therefore \quad Au'' + Bu' + Cu = 0 \ , \quad (7)$$
$$\text{and} \quad Av'' + Bv' + Cv = 0 \ . \quad (8)$$

Using equations (7) and (8) in the equation (6), we get

$$X'u' + Y'v' = R \ , \quad (9)$$

Solving (4) and (9) by Cramers rule we get

$$X' = \frac{-vR}{uv' - u'v}, \quad Y' = \frac{uR}{uv' - u'v} \ ,$$

integrating we get

$$X = -\int \frac{vR}{uv' - u'v} dx, \quad Y = \int \frac{uR}{uv' - u'v} dx \ .$$

putting X and Y in the equation (3), we get

$$\text{P. I.} = -u\int \frac{vR}{uv' - u'v}dx + v\int \frac{uR}{uv' - u'v}dx \ ,$$

$$\Rightarrow \text{P. I.} = -u\int \frac{vR}{W}dx + v\int \frac{uR}{W}dx \ ,$$

where $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$. Finally solution is given by

$$y = C.F. + P.I.$$

Example 1.22: Solve by using method of variation of parameter

$$\frac{d^2y}{dx^2} + y = tanx$$

Solution: Symbolic form of given equation is

$$(D^2 + 1)y = tanx \ ,$$

$$\text{A. E. is} \ \ D^2 + 1 = 0 \Rightarrow D = \pm i \ ,$$

$$\therefore \quad \text{C. F.} = c_1 cosx + c_2 sinx = c_1 u + c_2 v \ ,$$

where $u = cosx$, $v = sinx$, $R = tanx$ and

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} cosx & sinx \\ -sinx & cosx \end{vmatrix} = cos^2x + sin^2x = 1$$

Now

$$\begin{aligned}
\text{P. I.} &= -u\int \frac{vR}{W}dx + v\int \frac{uR}{W}dx \ , \\
&= -cosx\int \frac{sinx \cdot tanx}{1}dx + sinx\int \frac{cosx \cdot tanx}{1}dx \ , \\
&= -cosx\int \frac{sin^2x}{cosx}dx + sinx\int sinxdx \ , \\
&= -cosx\int \frac{1-cos^2x}{cosx}dx + sinx(-cosx) \ , \\
&= -cosx\int (secx - cosx)dx - sinx \cdot cosx \ , \\
&= cosx[sinx - log(secx + tanx)] - sinx \cdot cosx \ , \\
&= -cosx \cdot log(secx + tanx) \ .
\end{aligned}$$

Complete solution is

$$y = \text{C. F.} + \text{P. I.} = c_1 cosx + c_2 sinx - cosx \cdot log(secx + tanx) \ .$$

Example 1.23: Solve by using method of variation of parameter

$$\frac{d^2y}{dx^2} + y = cosecx$$

Solution: Symbolic form of given equation is

$$(D^2+1)y = cosecx \ ,$$

$$\text{A. E. is } \ D^2 + 1 = 0 \Rightarrow D = \pm i \ ,$$

$$\therefore \quad \text{C. F.} = c_1 cosx + c_2 sinx = c_1 u + c_2 v \ ,$$

where $u = cosx$, $v = sinx$, $R = cosecx$ and

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} cosx & sinx \\ -sinx & cosx \end{vmatrix} = cos^2x + sin^2x = 1$$

Now

$$\text{P. I.} = -u \int \frac{vR}{W} dx + v \int \frac{uR}{W} dx \ ,$$

$$= -cosx \int \frac{sinx \cdot cosecx}{1} dx + sinx \int \frac{cosx \cdot cosecx}{1} dx \ ,$$

$$\text{P. I.} = -\,cosx\int dx + sinx\int cotxdx\ ,$$
$$= -\,xcosx + sinx\cdot log(sinx)\ .$$

1.2.1 Ordinary Simultaneous Differential Equations

Introduction: The general form of Simultaneous differential equations of first order and of the first degree containing three variables is

$$P_1dx + Q_1dy + R_1dz = 0\ , P_2dx + Q_2dy + R_2dz = 0\ , \qquad (1)$$

where the coefficients are function of x, y, z. Solving these equations simultaneously, we get,

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{P_2R_1 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1}\ ,$$

OR

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\ . \qquad (2)$$

Thus simultaneous equations (1) can always be put in the form (2).

1.2.2 Method of Solving Simultaneous Linear Differential Equations.

Method I: Suppose that any two fractions, directly integrated then after integration we find an integral. Same procedure we apply for other two fractions. The two integrals so obtained form the complete solution. Sometimes first integral may be used to simplify the other two fractions.

Example 1.27: Solve

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Solution: From first two fractions, we have

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow xdx = ydy, \text{on integration, we get } \; x^2 = y^2 + c_1 \; \text{ or}$$

$$x^2 - y^2 = c_1 \; .$$

Similarly, from second and third fraction, we have

$$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow ydy = zdz, \text{on integration, we get } \; y^2 = z^2 + c_2 \; \text{ or}$$

$$y^2 - z^2 = c_2 \; .$$

$\therefore$ Complete solution is

$$x^2 - y^2 = c_1, y^2 - z^2 = c_2 \; .$$

Method II: We have,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR} \; .$$

Choose l, m, n in such a way that $lP + mQ + nR = 0$, so that $ldx + mdy + ndz = 0$. If L. H. S. is an exact differential du (say) then $du = 0$ and $u = c_1$ is one part of the complete solution. Here l, m, n are known as multipliers. This method may be repeated to get another integral by choosing new set of multipliers l', m', n'.
Note: Sometimes one integral find by using Method I and second integral by using Method II.

Example 1.30:

Solve

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Solution: Choosing l, m, n as multipliers, we get

$$\text{Each Fraction} = \frac{ldx + mdy + ndz}{lmz - lny + mnx - lmz + nly - mnx} = \frac{ldx + mdy + n}{0}$$

$$\therefore \quad ldx + mdy + ndz = 0 \ \text{ on integration, we get } \ lx + my + nz =$$

Again, choosing x, y, z as multipliers, we get

$$\text{Each Fraction} = \frac{xdx + ydy + zdz}{mxz - nxy + nxy - lyz + lyz - mxz} = \frac{xdx + ydy + zd}{0}$$

$$\therefore \quad xdx + ydy + zdz = 0 \ \text{ on integration, we get } \ x^2 + y^2 + z^2 = c$$

$\therefore$ complete solution is

$$lx + my + nz = c_1, \quad x^2 + y^2 + z^2 = c_2 \ .$$

1.2.3 Total differential equations:

An equation of the form

$$Pdx + Qdy + Rdz = 0 \ , \tag{1}$$

where P, Q, R are functions of x, y, z is called total differential equation.
Sometimes equation (1) can be directly integrable, if there exists a function $s(x, y, z)$ whose total differential ds is equal to L.H.S. of (1), otherwise we find the condition for which equation (1) be integrable.

1.2.4 Necessary condition for Integrability of total differential equation

Let the total differential equation be

$$Pdx + Qdy + Rdz = 0 \ . \tag{2}$$

Let $s(x, y, z) = c$ be a solution of equation (2). Then total differential ds must be equal $Pdx + Qdy + Rdz$, but, we know that

$$ds = \frac{\partial s}{\partial x}dx + \frac{\partial s}{\partial y}dy + \frac{\partial s}{\partial z}dz = 0 \ . \tag{3}$$

On comparing (2) and (3), we have

$$\frac{\frac{\partial s}{\partial x}}{P} = \frac{\frac{\partial s}{\partial y}}{Q} = \frac{\frac{\partial s}{\partial z}}{R} = \mu(x, y, z)(\text{say}) \ ,$$

so that

$$\frac{\partial s}{\partial x} = \mu P \ , \quad \frac{\partial s}{\partial y} = \mu Q \ , \quad \frac{\partial s}{\partial z} = \mu R \ . \tag{4}$$

Now from first two equations, we have

$$\frac{\partial^2 s}{\partial y \partial x} = \frac{\partial}{\partial y}(\mu P) = P\frac{\partial \mu}{\partial y} + \mu\frac{\partial P}{\partial y} \ \text{ and}$$
$$\frac{\partial^2 s}{\partial x \partial y} = \frac{\partial}{\partial x}(\mu Q) = Q\frac{\partial \mu}{\partial x} + \mu\frac{\partial Q}{\partial x} \ .$$

Since

$$\frac{\partial^2 s}{\partial y \partial x} = \frac{\partial^2 s}{\partial x \partial y} \ , \quad \text{we have} \quad P\frac{\partial \mu}{\partial y} + \mu\frac{\partial P}{\partial y} = Q\frac{\partial \mu}{\partial x} + \mu\frac{\partial Q}{\partial x} \ ,$$

$$\Rightarrow \quad \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y} \ , \tag{5}$$

similarly,

$$\mu\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial z} \ , \tag{6}$$

and

$$\mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial x} \ . \tag{7}$$

Multiplying (5), (6) and (7) by R, P, and Q respectively and adding, we get

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \ , \tag{0.1}$$

$$\text{i. e.} \quad \begin{vmatrix} P & Q & R \\ P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = 0 \ . \qquad (8)$$

This is the required condition for integrability of equation (1).

1.2.5 The Condition for exactness. of $Pdx + Qdy + Rdz = 0$.

Consider the total differential equation

$$Pdx + Qdy + Rdz = 0 \ , \qquad (1)$$

The equation (1) is said to be exact, if the following three conditions are satisfied

$$\left(\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\right) \ , \left(\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}\right) \text{ and } \left(\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}\right) \ . \qquad (2)$$

Note that when conditions (2) are satisfied, the condition for integrability of $Pdx + Qdy + Rdz = 0$, namely,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \ , \qquad (3)$$

is also satisfied, for each term of (3), vanishes identically.

1.2.6 Methods of Solving total differential equations:

1.2.6 (a) Method I: Method of Inspection

If the given equation is exact then the coefficients of P, Q and R in (8) are zero. In this case given equation is directly integrable after regrouping the terms of the equation.

Example 1.40: Solve

$$(x-y)dx - xdy + zdz = 0$$

Solution: We have,

$$P = x - y \ , \quad Q = -x \ , \quad R = z \ ,$$

$$\therefore \quad \frac{\partial P}{\partial y} = -1 \ , \frac{\partial P}{\partial z} = 0 \ , \frac{\partial Q}{\partial x} = -1 \ , \frac{\partial Q}{\partial z} = 0 \ , \frac{\partial R}{\partial x} = 0 \ , \frac{\partial R}{\partial y} = 0 \ ,$$

The condition of integrability is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$
$$= (x-y)(0-0) - x(0-0) + z(-1+1) = 0 \ ,$$

satisfied here, hence given equation is exact. Now rearranging the terms of the given equation, we have

$$xdx - (ydx + xdy) + zdz = 0 \ ,$$

on integration, we get solution as

$$x^2 - 2xy + z^2 = c_1 \ .$$

1.2.6 (b) Method II: Treating One Variable as Constant

1. Verify the condition of integrability.
2. Check, which two terms in the given equation can be easily integrable.
3. Take the variable in the remaining term as constant viz for our convenience let two terms $Pdx + Qdy = 0$ be easily integrable, then we take the third variable $z =$ constant, so that we get $dz = 0$.
4. Put $dz = 0$ in the given equation and then integrate, add $\phi(z)$ as constant of integration.
5. Differentiate the equation obtained in step (4) with respect to x, y ,z and compare it with the given equation to find $\phi^{'}(z)$, which on integration gives the value of $\phi(z)$.

1. Substitute this value of $\phi(z)$ in the relation obtained in step (4), gives the complete solution of the given equation.

Example 1.45: Solve

$$(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0$$

Solution: We have,

$$P = 2x^2 + 2xy + 2xz^2 + 1 \ , \quad Q = 1 \ , \quad R = 2z \ ,$$
$$\therefore \quad \frac{\partial P}{\partial y} = 2x \ , \frac{\partial P}{\partial z} = 4xz \ , \frac{\partial Q}{\partial x} = 0 \ , \frac{\partial Q}{\partial z} = 0 \ , \frac{\partial R}{\partial x} = 0 \ , \frac{\partial R}{\partial y} = 0 \ ,$$

The condition of integrability is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$
$$= (2x^2 + 2xy + 2xz^2 + 1)(0 - 0) + 1(0 - 4xz) + 2z(2x - 0) = 0 \ ,$$

satisfied here, hence given equation is integrable. Take $x =$constant.

$\therefore\ dx = 0$ and the given equation becomes

$$dy + 2zdz = 0\ , \quad \text{on integrating, we get} \quad y + z^2 = \phi(x)\ . \qquad (1)$$

Now on differentiating (1), we get

$$dy + 2zdz = \phi'(x)dx \Rightarrow\ dy + 2zdz - \phi'(x)dx = 0\ , \qquad (2)$$

comparing it with given equation, we get

$$\begin{aligned} -\phi'(x) =& 2x^2 + 2xy + 2xz^2 + 1\ , \\ =& 2x^2 + 2x(y + z^2) + 1 \\ =& 2x^2 + 2x\phi(x) + 1\ \ (\because \text{by}(1))\ , \end{aligned}$$

$$\therefore\quad \phi'(x) + 2x\phi(x) = -\,(2x^2 + 1)\ ,$$

which is linear equation in $\phi(x)$, hence its solution is given by

$$\phi(x) \cdot \text{I. F.} = \int \text{I. F.} \cdot Qdx + c\ , \qquad (3)$$

$$\text{where}\quad \text{I. F.} = e^{x^2}\ ,$$

Hence equation (3) becomes

$$\begin{aligned}
\phi(x)\cdot e^{x^2} =& c - \int (2x^2+1)e^{x^2}dx = c - \int (2x^2 e^{x^2})dx - \int e^{x^2}dx\ ,\\
=& c - \int x \cdot 2xe^{x^2}dx - \int e^{x^2}dx = c - xe^{x^2}\ ,\\
\therefore\quad \phi(x) =& ce^{-x^2} - x\ .
\end{aligned}$$

putting this value of $\phi(x)$ in equation (1), we get general solution as

$$y + z^2 = ce^{-x^2} - x \Rightarrow \quad (x + y + z^2)e^{x^2} = c\ .$$

1.2.7 Geometrical interpretation of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$:
Consider the equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\ . \tag{1}$$

From three dimensional coordinate geometry, it is know that the direction cosines of the tangent to a curve are proportional to dx, dy, dz. The differential equation (1), therefore, express the fact that the direction cosines of the tangent to the curve at that point are proportional to P, Q, R. Let the solution of equation (1) is given by $u = c_1$ and $v = c_2$. Then we observe that the solution represents the curve of intersection of the surfaces $u = c_1$ and $v = c_2$. Since c_1 and c_2 takes infinite many values, we get doubly infinite number of such curves.

1.2.8 Geometrical Interpretation of $Pdx + Qdy + Rdz = 0$:

Let the total differential equation be

$$Pdx + Qdy + Rdz = 0 \ , \tag{1}$$

let $f(x, y, z) = c$ be integral of the equation (1).

$$\therefore \quad \frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = 0 \ , \tag{2}$$

comparing (1) and (2), we get

$$\frac{\frac{df}{dx}}{P} = \frac{\frac{df}{dy}}{Q} = \frac{\frac{df}{dz}}{R} \ .$$

But, $\frac{df}{dx}$, $\frac{df}{dy}$, $\frac{df}{dz}$ are proportional to direction cosines (d.c.s.) of normal to the surface $f(x, y, z) = c$ at point $P(x, y, z)$ on it.

Hence, equation (1) represents a one-parameter family of parallel planes in xyz-plane such that d.c.s. of normal at any point $P(x, y, z)$ on the surface are proportional to the values of P, Q, R at that point.

1.2.9 Geometrical Relation Between Total Differential Equation And Simultaneous Differential Equations

Let the total differential equation be

$$Pdx + Qdy + Rdz = 0 \ , \tag{1}$$

and system of simultaneous differential equations associated with (1) namely

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \ . \tag{2}$$

To examine the relationship between the solution set of equations (1) and (2), let us consider the following total differential equation

$$dx + dy + dz = 0 \ , \tag{3}$$

on integrating, we get

$$x + y + z = c \ , \ -\infty < c < \infty \ . \tag{4}$$

This is a one-parameter family of parallel planes in the *xyz*-plane. Now system of simultaneous differential equations associated with (3) is

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1} \ . \tag{5}$$

The solution set of this equation is a two-parameter family of straight lines

$$x - y = c_1 \ \text{ and } \ y - z = c_2 \ , \tag{6}$$

we know that a family of straight lines intersect a family of planes orthogonally if the direction cosines of the family of straight lines are same as the direction cosines of normal to the family of planes.

Now equation (6) can be written as

$$\frac{x - c_1}{1} = \frac{y}{1} = \frac{z + c_2}{1}$$

$\therefore$ d.c.s. of family of lines (6) are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Also, d.r.s. of the normal to the family of plane (4) are (1, 1, 1) so that the d.c.s. of normal to plane (4) are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

$\therefore$ The family of straight (6) intersect the family of plane (4) orthogonally.

Hence, the locus of total differential equations
$Pdx + Qdy + Rdz = 0$ is orthogonal to the locus of simultaneous differential equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

THANK YOU